Magnetic Field Measurement in the Solar Atmosphere Problem Set

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1 Zeeman effect in L-S coupling scheme

When an atom is placed in a uniform external magnetic field $\boldsymbol{B}$, the total Hamiltonian can be written as

$$H = H_{\text{bohr}} + H_{\text{fs}} + H_B \tag{1}$$

where H_{bohr} is the sum of electron kinetic energy and Coulomb potential, H_{fs} is the fine structure Hamiltonian. H_B is the Hamiltonian introduced by external magnetic field

$$H_B = \frac{e_0}{2m_e}\left(\boldsymbol{L} + 2\boldsymbol{S}\right)\cdot\boldsymbol{B} + \frac{e_0^2}{8m_e}\left(\boldsymbol{B}\times\boldsymbol{r}\right) \tag{2}$$

where $\boldsymbol{L}$ and $\boldsymbol{S}$ are orbital and spin angular momentum. In this problem, we assume $H_B \ll H_{\text{bohr}}$ and $H_B \ll H_{\text{fs}}$ so that we can treat the first two terms as an "unperturbed" part of the Hamiltonian and the H_B as the "perturbation".

a) The first term in H_B is the well-known Zeeman effect term, and the second term is the diamagnetic term. Using order of magnitude analysis to show that in the regime of solar magnetic fields ($B < 5000$ G) the second term is negligible compared to the first term. So that we can write

$$H_B = \frac{e_0}{2m_e}\left(\boldsymbol{L} + 2\boldsymbol{S}\right)\cdot\boldsymbol{B} \tag{3}$$

b) (Landé factor and Energy level splitting) The L-S coupling scheme assumes n, l, j, m are all "good" quantum numbers in this problem, so that the first-order energy perturbation caused by Zeeman effect can be evaluated as

$$\Delta E = \langle nljm|H_B|nljm\rangle = \frac{e}{2m_e}\boldsymbol{B}\cdot\langle nljm|(\boldsymbol{J}+\boldsymbol{S})|nljm\rangle \tag{4}$$

Consider a semi-classical picture of the orbital coupling[1], the spin angular momentum $\boldsymbol{S}$ rotates rapidly around the total angular momentum $\boldsymbol{J}$, so that the effective (time-averaged) component of

[1] One can also use the projection theorem, which is a special case of the Wigner-Eckart theorem, to obtain the same answer.

$\boldsymbol{S}$ is its projection to $\boldsymbol{J}$, i.e.

$$\boldsymbol{S}_{\text{eff}} = \frac{\boldsymbol{S}\cdot\boldsymbol{J}}{J^2}\boldsymbol{J} \tag{5}$$

Choosing the direction of magnetic field as z axis, and Using Eq.4 and 5 to show that the energy perturbation is

$$\Delta E = g_j \mu_B m_j B,\ m_j = 0, \pm 1, \cdots \pm j \tag{6}$$

where g_J is the Landé factor

$$g_j = 1 + \frac{j(j+1) + s(s+1) - l(l+1)}{2j(j+1)} \tag{7}$$

and μ_B is the Bohr magneton

$$\mu_B = \frac{e_0 \hbar}{2m_e} \tag{8}$$

using Eq. 7 to calculate and compared the Landé factors of the following orbitals $^4\mathrm{D}_{1/2}$, $^2\mathrm{P}_{1/2}$,$^6\mathrm{F}_{1/2}$, and $^6\mathrm{D}_{1/2}$.

c) (Effective Landé factor) The Landé factor of the upper and lower energy level of the transition is not necessarily the same. In order to evaluate the sensitivity of a spectral line to the magnetic field, we can define an effective Landé factor as

$$g_{\text{eff}} = \frac{1}{2}(g + g') + \frac{1}{4}(g - g')(j(j+1) - j'(j'+1)) \tag{9}$$

larger g_{eff} indicates that the spectral line is more sensitive to the magnetic fields. Calculate and compared the effective Landé factors of the spectral lines given in Table 1.

Spectral Line	j	j'	g_j	g'_j
Fe I 6301 Å	2	2	1.84	1.50
Fe I 6302 Å	1	0	2.49	0
Fe I 6173 Å	1	0	2.50	0
Fe I 15648 Å	1	1	3	2.95

Table 1: Atomic data of commonly used Fe I lines in solar magnetic field measurements.

d) (Splitting of spectral lines) The selection rule of electric dipole transition requires $\Delta m_j = 0, \pm 1$. Show that the difference of the photon wavelength between the one transition from $|nljm_j\rangle$ to $|n'l'j'm'_j\rangle$ and the unperturbed transition is

$$\Delta\lambda = \Delta\lambda_B (g_j m_j - g'_j m'_j) \tag{10}$$

where g_j and g'_j are Landé factors of the upper and lower energy level. And

$$\Delta\lambda_B = \frac{e_0 B}{4\pi m_e c}\lambda_0^2 \tag{11}$$

if you want to increase the splitting wavelength $\Delta\lambda$ in real measurements of the solar magnetic field, which lines should you use, ultraviolet, visible or infrared lines?

e)(Polarization of the splitting spectral lines) The photons emitted from splitting energy levels are polarized, when $\Delta m_j = -1$, the light are right-handed circular polarized, which is called a σ^- line. $\Delta m_j = +1$ photons are left-handed circular polarized, which is called a σ^+ line. $\Delta m_j = 0$ photons are linearly polarized and are called π lines. When the line of sight (LOS) of the observer is parallel to the direction of the magnetic field, only σ lines can be observed. When the LOS is perpendicular to the magnetic field, all the σ and π lines can be observed.

Now considering the Na I D_2 $\left({}^2P_{3/2} \rightarrow {}^2P_{1/2}\right)$ line, how many lines do you expect to observe when the external magnetic field is presented? Calculate the wavelength difference between the unperturbed transition and write down the polarization state of each splitting spectral line.

f)(Weak field criteria) At the beginning we assumed that $H_B \ll H_{\text{fs}}$, which means the formalism we just developed can only be applied in weak field conditions. To give an order of magnitude estimation of the "weak" field limit, calculate the splitting energy of the $2p$ orbital of a hydrogen atom and compare the results with the fine structure energy of hydrogen atom:

$$\Delta E_{\text{fs}} = -\alpha^2 \frac{Rhc}{n^3}\left(\frac{1}{j+1/2} - \frac{3}{4n}\right) \tag{12}$$

where α is the fine structure constant and R is the Rydberg constant.

g)(Blended line profiles/Detection limit) Spectral lines in the solar atmosphere are not δ functions. They are broadened by different processes like the thermal motions of the particles and pressure broadening caused by interaction with ambient particles. Assume that we have a spectrometer with infinite spectral resolution, we still require the splitting between the perturbed and unperturbed line profile larger than the full width at half maximum (FWHM) of the spectral line. Otherwise the profiles will be blended and we cannot measure the splitting from two blended profiles.

Using the atomic data of Fe I 6301 Å line shown in Table 1 and assuming FWHM$\sim$ 0.2 Å, calculate the minimal magnetic field that can be measured.

2 Hands-on real magnetic field inversion

Go to the Milne-Eddington Simulator website which synthesizes the Stokes parameters I, Q, U, V from Milne-Eddington approximations. The Stokes parameters are another way to represent the polarization state of the light. Generally speaking, I measures the total intensity, Q and U measure the linear polarization, and V measures the circular polarization. Now you can start play with the simulator:

a) Change the magnetic field strength. How do the line profiles change (e.g. values and position of the peaks and valleys)? Why? Can you distinguish the splitting line profiles in Stokes I?

b) Change the field inclination. What do you find with the Stokes parameters? Can you explain why? Do the same thing when changing the field azimuth.

c) Change the line width. Describe how do the Stokes parameters change and explain them.

d) Compare the Stokes I and V profiles. Note that the Stokes V behaves like the derivative of the Stokes I, why?

In real observations, we actually fit the observed Stokes profiles with these parameters, which is called Milne-Eddington inversion. The magnetogram shown in Figure 1 is obtained using the Milne-Eddington inversion.

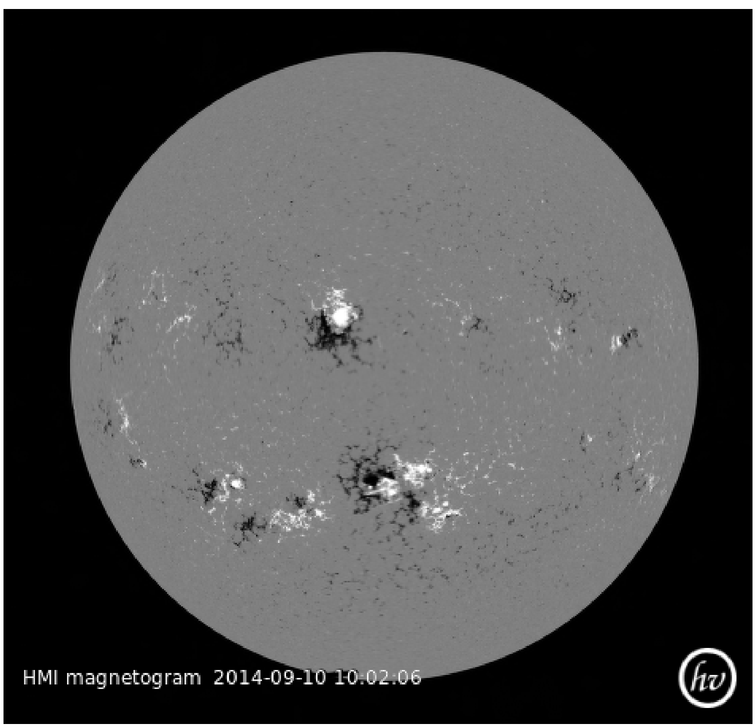

Figure 1: Full-disk LOS magnetogram observed by Helioseismic and Magnetic Imager (HMI) on board Solar Dynamics Observatory (SDO) using the Milne-Eddington inversion of the Fe I 6173 Å line.

References

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