General Relativity

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BH physics

QFT

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Prerequisite: classical physics

- Electrodynamics $\begin{cases} \text{4-vector} \\ \text{SR.} \end{cases}$

- Action. Lagrangian. Euler-Lagrangian eq.

- Math: $\begin{cases} \text{Multivariable analysis} \\ \text{Diff eq.} \end{cases}$

Grading: 30% HW + 70% final

Reference Books:

I. 广义相对论 (2018)

II. Lecture Notes.

III. S. Carroll "Space & Geometry" 2013.

IV. J. Hartle "Gravity" ..

V. B. Schutz "A First course in GR"

VI. R. Wald "General relativity" 1984

VII. MTW "Gravitation" (1973)

VIII. S. Weinberg "Gravity & Cosmology" 1972

IX. 梁灿彬. 周彬. 微分几何入门与广义相对论

(2nd ed.)

GR $\rightarrow$ A relativistic theory of gravity

Gravity
- universal mass, energy
- attractive $\rightarrow$ "charge" $\rightarrow$ graviton spin 2
- long range $F \propto \frac{1}{r^2} \longleftrightarrow F_{em} \sim \frac{1}{r^2}$

$\downarrow$ γ spin 1

EM
weak
strong

$\}$ $\rightarrow$ Standard model (QFT - gauge theory)

$\rightarrow$ Quantum Gravity?

(unification).

- weakest force $p-p$ $\frac{F_g}{F_{em}} \sim 10^{-36}$

Gravity: Universa

(Cosmology - galaxy - solar system - star - Moon & Earth - Earth)

History

- Galileo. $\rightarrow$ Equivalence principle.
- Kepler
- Newton. tidal force.

- Laplace. Poisson. Hamilton
  perturbation theory 19th.

- Mercury precession

- SR Maxwell theory. $\overset{\times}{\longleftarrow}$ relativistic principle

  → ether?

Einstein 1905 SR.

↓

1908 SR ⟺ gravity <u>Inconsistency</u>

- Instanteneous. interaction
- Not invariant under Lorentz transform

↓

1915. GR. Einstein field theory

1919. light deflection.

- gravitational red shift.

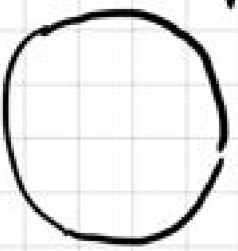
2015. GW (LIGO)

Direct detection.

Why GR?

<u>SR</u> $v \sim c$ relativistic effect

M. R.



escape velocity.

$$v \sim \sqrt{\frac{2GM}{R}}$$

$v \sim c$? $\gamma_g \sim \frac{2GM}{Rc^2}$ dimensionless.

$\gamma \sim 0.5$ BH

~ 0.1 NS.

1. Earth $M_\oplus$ $R_\oplus$ $\gamma_g \sim 10^{-5}$

$\longrightarrow$ GPS.

gravitional redshift

2. Sun. $M_\odot$ $R_{\mu\nu}$ $r_g \sim 10^{-6}$

3. Cosmology. $\rightarrow$ Evolution.

Einstein's theory about spacetime & gravity.

Gravity = Spacetime ("curved").

1. Einstein Eq.

$$\underbrace{R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R}_{\text{geometry}} = 8\pi G \underline{T_{\mu\nu}}$$

$\downarrow$ energy-momentum tensor

Matter - geometry.

2. Geodesic eq. "point particle" in gravity field.

$$\frac{d^2 x^\mu}{d\lambda^2} - \Gamma^\mu_{\sigma\rho} \frac{dx^\sigma}{d\lambda} \frac{dx^\rho}{d\lambda} = 0.$$

Outline:

1. SR: geometry
2. Gravity & SR. → GR.
3. Differential geometry manifold.

   vector field ..
4. Einstein Eq.
5. Schwarzchild solution → BH.
6. Kerr solution → BH.
7. GW. linearlized gravity in weak field approximation.

Special Relativity.

Newtonian mechanics. → Inertial frame.

↳ Galileo transform

Galileo - Newton

space & <u>time</u> absolute.

$$t' = t$$

$$\vec{x}' = H\vec{x} + \vec{v}t + \vec{x}_0$$

↓

rot

Relavitivistic principle

All physics laws are inv in all inertial frames.

↓

· Maxwell theory?

↓

not inv under Galileo transform.

· Einstein.

Relativistic principle ✓

light speed inv principle.

→ Lorentz transform.

$R \oplus R^3 \rightarrow R^{1,3}$ "Minkowski" spacetime

↳ General spacetime in GR.

· Event point in space time

Spacetime {events}.

"world line" trajectory of a point particle

$\begin{cases} \text{massless.} \\ \text{massive.} \end{cases}$

Event. spacetime. world line

→ coordinate irrelavant.

→ coordinates: Event (t, x, y, z)

x^μ x^0 x^1 x^2 x^3.

world line $x^\mu(\lambda)$.

↳ parameterize the curve.

Geometry. ← distance.

$$ds^2 = \sum_{ij} g_{ij}\, dx^i dx^j$$

g_{ij} ~ metric $\begin{cases} \text{ruler ?} \\ \text{clock ?} \end{cases}$

R^2.

1) Cartesian:

$$ds^2 = dx^2 + dy^2$$

$$g_{ij} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

2) polar coordinate.

$$ds^2 = d\rho^2 + \rho^2 d\phi^2$$

$$g_{ij} = \begin{pmatrix} 1 & 0 \\ 0 & \rho^2 \end{pmatrix}$$

however ds^2 is coordinate indep

g_{ij} cannot tell you if a spacetime is curved! (coordinate depend).

$R^{1,3}$ in Cartesian coordinates.

$$ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu$$

· Summation rules.

$$\cdot\ \eta_{\mu\nu} = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$$

$$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2 \quad (c=1).$$

$$(x^0 = ct.\ x^1 = x.\ x^2 = y.\ x^3 = z).$$

convention.

1. Summation rules

2. $\mu . \nu . \sigma \ldots$
   $0.1.2.3$

3. $i . j . k . m$
   $1.2.3$ (spatial).

4. $\eta_{\mu\nu}$ $(-1, 1, 1, 1)$. (East coast).

· Lorentz transformation.

$$x^{\mu'} = \Lambda^{\mu'}{}_{\nu} x^{\nu}$$

s.t ds^2 inv.

$$\eta_{\mu'\nu'} dx^{\mu'} dx^{\nu'} = \eta_{\mu\nu} dx^{\mu} dx^{\nu}$$

$$\eta_{\mu'\nu'} \Lambda^{\mu'}{}_{\mu} \Lambda^{\nu'}{}_{\nu} = \eta_{\mu\nu}$$

Spacetime in SR

Minkowski spacetime (flat)

Inertial frame

→ coordinate system.

$R^3 \to \delta_{ij}$

$\delta_{ij} \Lambda^i{}_k \Lambda^j{}_l = \delta_{kl} \Rightarrow \Lambda^T \Lambda = I \to O(3)$

$\det \Lambda = 1 \Rightarrow SO(3)$ group → non-Abelian

⇒ 3 generators.

Group {element}.

product $\to$ closed.

identity element.

$R^{1,3}$ $\Lambda^T \eta \Lambda = \eta$

$\{\Lambda\} \to O(1,3)$

$(\Lambda^0{}_0)^2 = 1 + (\Lambda^i{}_0)^2 \geq 1.$

$\to |\Lambda^0{}_0| \geq 1$

if. $\Lambda^0{}_0 \geq 1$. $\det(\Lambda) = 1 \to SO(1,3)$

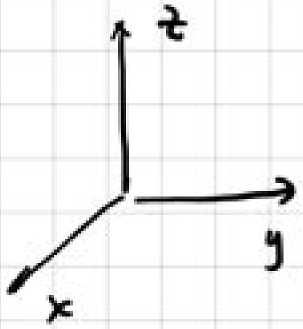
proper Lorentz group

(keep time direction)

other transform?

- time-reversal
- $x^\mu \to x^\mu + a^\mu$ $\Rightarrow$ Poincare group.
- $x^\mu \to - x^\mu$

x^i : {rotations} $\in SO(1,3)$



$(x, y) \to$ rotation.

(3D space)

$\downarrow$

$(t, x) \xrightarrow{?}$ rotation?

(Lorentz boost).

Example: for a 2×2 Λ.

$$\Lambda = \begin{pmatrix} \cosh\phi & -\sinh\phi \\ -\sinh\phi & \cosh\phi \end{pmatrix} \longleftarrow \eta_{\mu'\nu'}\Lambda^{\mu'}{}_{\sigma}\Lambda^{\nu'}{}_{\rho} = \eta_{\sigma\rho}.$$

$(0,0)$ component RHS $= -1$

$$LHS = \eta_{00}\Lambda^0{}_0\Lambda^0{}_0 + \eta_{11}\Lambda^1{}_0\Lambda^1{}_0$$

$$= -\cosh^2\phi + \sinh^2\phi = -1$$

rapidity

$x^\mu = \Lambda^\mu{}_\nu x^\nu$

actually

$$t' = t\cosh\phi - x\sinh\phi$$
$$x' = -t\sinh\phi + x\cosh\phi$$

$$\begin{cases} v = \tanh\phi \\ \gamma = \dfrac{1}{\sqrt{1-v^2}} = \cosh\phi \end{cases}$$

similiarly

every $(t, x^i) \rightarrow$ "rotation."

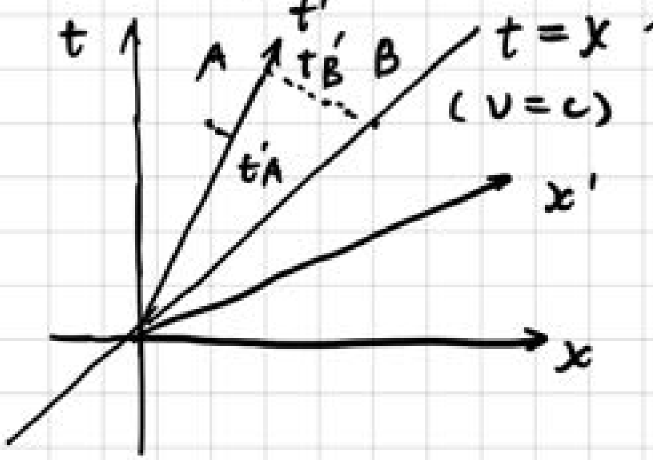
$C_4^2 = 6$ generators

3 for rotation

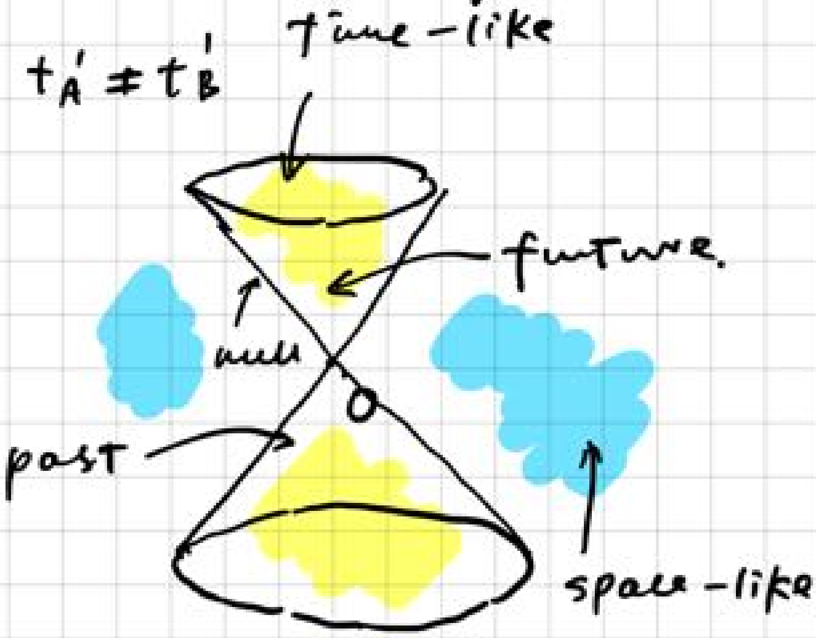
3 for Lorentz boost (also "rotations" in Minkowski space)

$$\underbrace{\Delta s^2}_{\text{interval}} = \eta_{\mu\nu} \Delta x^\mu \Delta x^\nu$$

· loss of simultaneity $\rightarrow$ c inv. $\Rightarrow$ light cone.



$t'_A \neq t'_B$



"Causality"

causal structure.

· vector

$\hat{a} \to R^{1,3}$

$\vec{a} \to R^{3}$

expansion. $\vec{a} = a_i \vec{e}_i \quad (i = 1, 2, 3)$

$\hat{a} = a^{\mu} \hat{e}_{\mu} \quad \{\hat{e}_{\mu}\} \quad (\mu = 0, 1, 2, 3)$

$\hat{a}$ coordinate independent

$$\left\{ \begin{array}{l} \hat{a} + \hat{b} \\ \alpha \hat{a} \end{array} \right. \longrightarrow \text{vector space.}$$

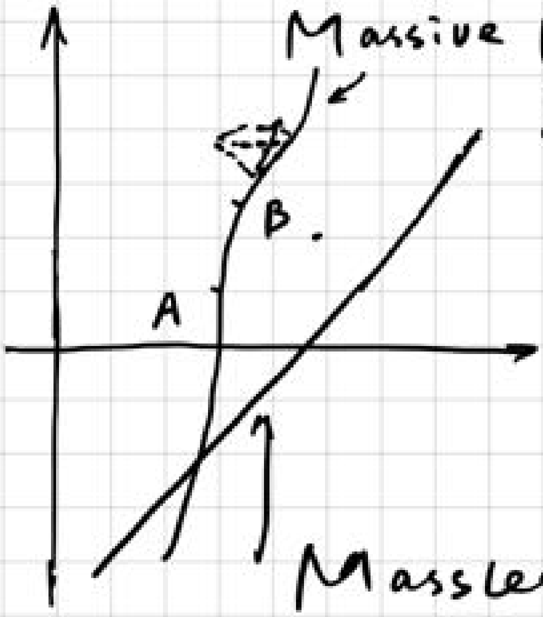
inner product?

$$\underbrace{\hat{a} \cdot \hat{b}}_{\downarrow \ \text{scaler}} = \eta_{\mu\nu} a^{\mu} b^{\nu}$$

$$\| \hat{a} \|^2 = \eta_{\mu\nu} a^{\mu} a^{\nu} \quad \left\{ \begin{array}{l} < 0, \ \hat{a} \ \text{time-like} \\ > 0, \ \hat{a} \ \text{space-like} \\ = 0, \ \hat{a} \ \text{null / light like} \end{array} \right.$$

orthogonal: $\hat{a} \cdot \hat{b} = 0$

Massive particle. world line.



$x^\mu(\lambda)$

$\hat{v}$: time-like vector.

Massless particle

$\hat{v}$: null vector.

Kinematics:

1. Massive particle time like worldline.

$x^\mu(\lambda)$ λ: parametrization.

$$\Delta s^2 = \eta_{\mu\nu}\, dx^\mu(\lambda)\, dx^\nu(\lambda)$$

$$= \underbrace{\eta_{\mu\nu} \frac{dx^\mu}{d\lambda} \cdot \frac{dx^\nu}{d\lambda}}_{\text{time-like} < 0} d\lambda^2$$

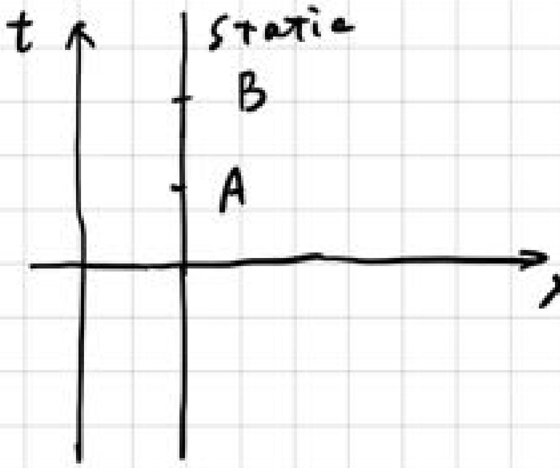
$$\text{def} \quad \Delta\tau = \int_{\lambda_A}^{\lambda_B} \sqrt{-\eta_{\mu\nu} \frac{dx^\mu}{d\lambda} \cdot \frac{dx^\nu}{d\lambda}}\, d\lambda.$$

$\lambda \to f(\lambda)$ $\Delta\tau$ inv. (re-parameterization inv)

· time-like curve

→ Local Inertial frame (LIF)

if the observer have a watch.



↓

from $A \to B$. the watch goes $\underbrace{\Delta\tau}$.

↓

proper time.

use proper time to parameterize the curve

$\to \lambda = \tau$ ($\lambda = a\tau + b$, affine parameterization)

$$\hat{u} = \frac{dx^\mu}{d\tau} \qquad \hat{u}\cdot\hat{u} = \frac{dx^\mu}{d\tau}\cdot\frac{dx^\nu}{d\tau}\eta_{\mu\nu}$$

$$= -\frac{d\tau^2}{d\tau^2} = -1$$

→ 4-velocity normalized.

static observer?

$$\begin{cases} x^i = \text{fixed} \\ t = \tau \end{cases} \qquad V^\mu_{obs} = (1,0,0,0).$$

$$V^{\mu'} = \gamma(1, -v^i)$$

$\hat{p}$ 4-momentum. $\hat{p} = m\hat{v}$

$$\hat{p}^2 = \eta_{\mu\nu} m^2 v^\mu v^\nu$$
$$= -m^2$$

LIF : $p^\mu = (m, 0, 0, 0)$.

Define $\rightarrow p = (E, p^i)$

$$\Rightarrow -E^2 + p^2 = -m^2.$$

That is $E^2 = p^2c^2 + m^2c^4$

evenly-accelerated motion (in LIF).

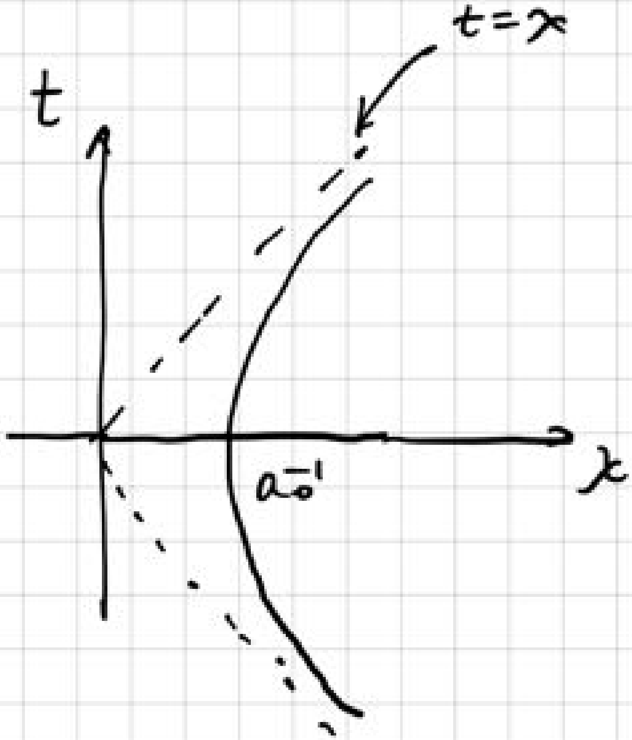
$$\begin{cases} t(\sigma) = a_0^{-1} \sinh\sigma \\ x(\sigma) = a_0^{-1} \cosh\sigma \end{cases}$$

$$\Rightarrow d\tau^2 = -dt^2 + dx^2 = (a_0^{-1})^2 (d\sigma)^2.$$

$$\Rightarrow \sigma \equiv a_0 \tau.$$

$$\Rightarrow \begin{cases} t(\tau) = a_0^{-1} \sinh(a_0\tau) \\ x(\tau) = a_0^{-1} \cosh(a_0\tau). \end{cases}$$

$$v^\mu = \frac{dx^\mu}{d\tau}$$



$$v^0 = \cosh a_0\tau$$

$$v^1 = \sinh a_0\tau$$

$$\hat{a} = \frac{d\hat{v}}{d\tau}$$

$$a^0 = a_0 \sinh a_0\tau$$

$$a^1 = a_0 \cosh a_0\tau$$

$$\|\hat{a}\|^2 = a_0^2$$

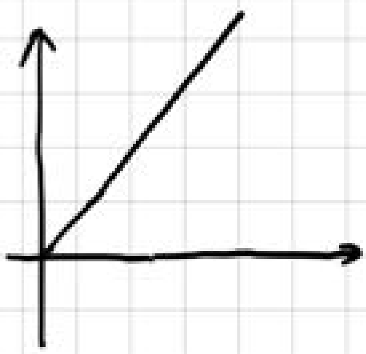
Massless particle:

$d\tau^2 = 0.$

· no proper time. but still should be parameterized. $x^\mu(\lambda)$

Example: photon travel along x-axis.

$\Rightarrow t = x$



$$\begin{cases} t = \lambda \\ x = \lambda \end{cases} ? \quad \begin{cases} t = \lambda^3 \\ x = \lambda^3 \end{cases} ?$$

Affine Parameterization!

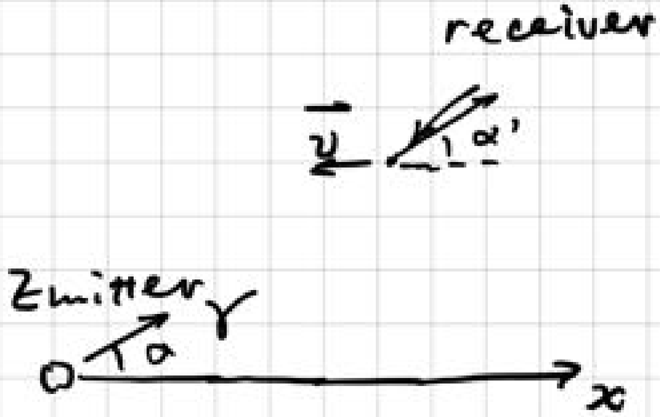
require: $\frac{du^\mu}{d\lambda} = 0$

$\Rightarrow \begin{cases} t = \lambda \\ x = \lambda \end{cases}$ ✓ (easier to handle)

↳ λ have no physical meaning.

$\begin{cases} \text{wave 4-vector } \hat{k} \\ \text{momentum 4-vector. } \hat{p} = \hbar \hat{k} \end{cases}$

$$p^\mu = (E, \vec{p}) = (\hbar\omega, \hbar\vec{k}) = \hbar k^\mu$$



1. Doppler effect.

$v > 0$ approach.

$v < 0$, leave.

S frame. $k^\mu = \frac{2\pi}{\lambda}(1, \cos\alpha, \sin\alpha, 0)$.

S' frame $k^{\mu'} = \frac{2\pi}{\lambda'}(1, \cos\alpha', \sin\alpha', 0)$.

$$k^{\mu'} = \Lambda^{\mu'}{}_{\mu} k^{\mu}$$

$$\downarrow$$

$$\begin{pmatrix} \gamma & -\beta\gamma & & \\ -\beta\gamma & \gamma & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$$

$$\to \frac{\lambda'}{\lambda} = \gamma(1-\beta\cos\alpha')$$

$$\tan\alpha = \frac{\tan\alpha'}{\gamma(1-\beta\cos\alpha')}$$

$$\Rightarrow f' = f \cdot \frac{\sqrt{1-v^2}}{1-v\cos\alpha'}$$

$\Delta f = fv$

Discussion

$\alpha' = 0$. $|v| \ll 1$. $\longrightarrow$ no $|v| \ll 1$?

$\Rightarrow f' = f(1+v)$ $\qquad f' = f\sqrt{\frac{1+v}{1-v}}$

$v > 0$ blue shift.

$v < 0$ red shift.

Doppler factor.

(Relativistic effect).

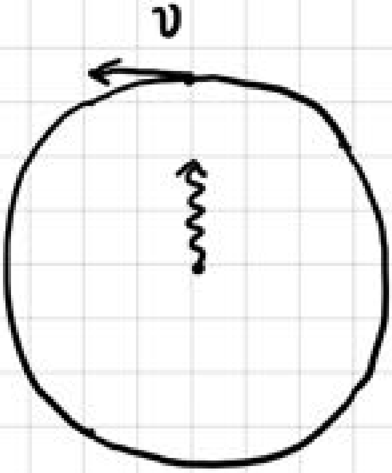
$\alpha' = \frac{\pi}{2}$ $\to$ transverse Dopper effect

$f' = f\sqrt{1-v^2}$ redshift.

use α instead of α'?

$$f' = f \frac{(1 + v\cos\alpha)}{\sqrt{1-\beta^2}}$$

$\alpha = \frac{\pi}{2}$ $\quad f' = f\gamma > f$ blue shift!



a receiver at the edge of a rotating plate.

Relativistic parallax?

$$\begin{cases} \cos\alpha = \frac{k_x}{\omega} \\ \cos\alpha' = \frac{k'_x}{\omega} \end{cases}$$

$$\Rightarrow \cos\alpha' = \frac{v + \cos\alpha}{1 + v\cos\alpha}$$

$$\tan\frac{\alpha'}{2} = \left(\frac{1-v}{1+v}\right)^{1/2} \tan\frac{\alpha}{2} = D^{-1}\tan\frac{\alpha}{2}$$

$v > 0 \quad \alpha' < \alpha \qquad \alpha = \frac{\pi}{2} \quad \cos\alpha' = v$

if $v \sim c \quad \alpha' \to 0$

the intensity magnification $\propto D^4$
(decrease in solid angle; frequency change)
compton scattering ~ 4 - momentum

conservation

Dynamics in relativity? (covariant)

$$\frac{d\vec{u}}{dt} = 0 \to \frac{d\hat{u}}{d\tau} = 0$$

inv in coordinate selection.

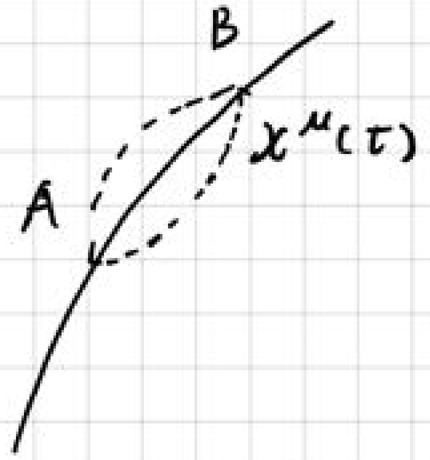
$$\vec{F} = m\vec{a} \ ? \qquad \vec{a} \to \hat{a} \quad \hat{a} = \frac{d\hat{u}}{d\tau}$$

$$\vec{F} \to \hat{f} \quad \text{4-force}$$

$\frac{v}{c} << 1 \to$ back to newtonian

$$\to \hat{f} = m\hat{a} \qquad \left(\begin{array}{l} \hat{u}\cdot\hat{u} = -1 \\ \Rightarrow \hat{a}\cdot\hat{u} = 0 \end{array} \right)$$

$$\to \hat{f}\cdot\hat{u} = 0$$



$$\tau_{AB} = \int_{\lambda_A}^{\lambda_B} \underbrace{\sqrt{-\eta_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}}}_{\text{Lagrangian?}} \, d\lambda$$

$\tau_{AB} \downarrow$ action

$\mathcal{L}(x^\mu(\lambda), \dot{x}^\mu(\lambda))$, $\dot{x} \to \frac{d}{d\lambda}$

Find $x^\mu(\lambda)$ s.t. $\delta\tau_{AB} = 0$

$\Rightarrow$ Euler-Lagrange Eq.

$$-\frac{d}{d\lambda}\left(\frac{\partial \mathcal{L}}{\partial\left(\frac{dx^\mu}{d\lambda}\right)}\right) + \frac{\partial \mathcal{L}}{\partial x^\mu} = 0$$

(Extremum, not necessary a minimum)

$$\Rightarrow \frac{d^2 x^\mu}{d\tau^2} = 0$$

in $R^{1,3}$. time-like worldline

$$\Rightarrow \frac{d\hat{u}}{d\tau} = 0$$

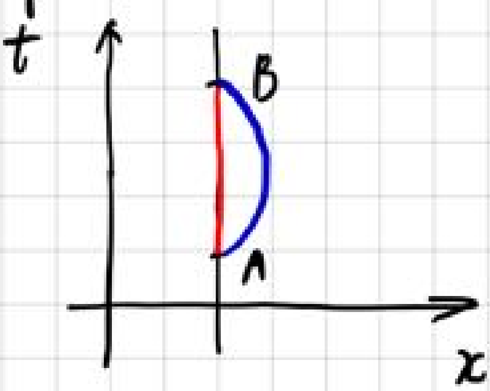
$\Rightarrow \Delta\tau \to$ maximum

Twin Paradox?

compare the proper time of two world lines:

A is older than B

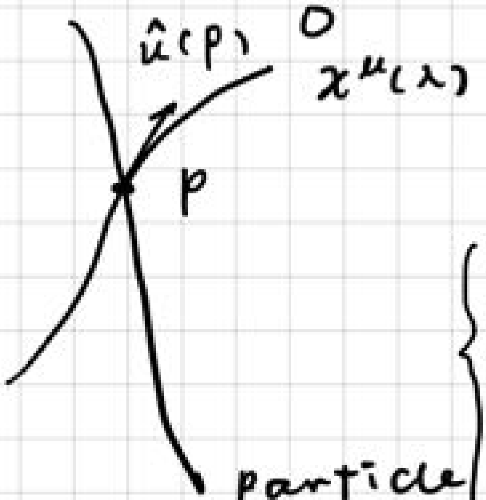
(maximum proper time)



Conclusion still holds in curved space-time $\dfrac{d^2x^\mu}{d\tau^2}=0 \longrightarrow$ Geodesic Eq.

Measure?

How to set up the Laboratary of the observer?



meet a particle at P.

$\hat{u}(p) \rightarrow$ time direction

particle $\begin{cases} (1, 0, 0, 0) \\ \\ \text{spatial direction} \\ \text{(coordinate)} \end{cases}$

$\Rightarrow \{\hat{e_\mu}\}\big|_0$ tetrod (标架场)

$$\begin{cases} \hat{e_\mu} = \hat{u} \\ \hat{e_u} \cdot \hat{e_i} = 0 \\ \hat{e_i} \cdot \hat{e_j} = \delta_{ij} \end{cases}$$ Lab set up!

Particle have a 4-momentum $\hat{p}$

w.r.t observor. $\hat{p}$. project to the tetrad (that's measurement!)

$$\begin{cases} E|_0 = -\hat{p}\cdot\hat{e}_\mu & \text{frame-independent} \\ p|_0 = \hat{p}\cdot\hat{e}_i & \leftarrow \text{some ambiguity?} \end{cases}$$

(degrees of freedom)

want we want:

if $\hat{e}_\mu$ fixed at $\tau=0$. $\hat{e}_\mu(t>0)$ is determined. (even if external force $\hat{f}$ (i.e. $\hat{a}$) exists)

$\Rightarrow$ Fermi-Walker transport

($\hat{e}_\mu$ fixed at $(\hat{a}, \hat{u})$ plane)

$$\begin{cases} \dfrac{d\hat{e}_\mu}{d\tau} = -(\hat{u}\cdot\hat{e}_\mu)\hat{a} + (\hat{a}\cdot\hat{e}_\mu)\hat{u} \\ \hat{e}_\mu|_{\tau=0} \end{cases}$$

$\hat{a}=0 \rightarrow \dfrac{d\hat{e}_\mu}{d\tau}=0$ totally fixed!

Energy of a moving particle

observer: $E|_o = m\gamma$

$\hat{p} = m\hat{u}$ $\quad \hat{u}^\mu|_p = (1, 0, 0, 0)$

$\hat{u}^\mu|_o = \gamma(1, \vec{v})$

from moving particle:

$\hat{p} = m(1, 0, 0, 0)$

$\hat{e}|_o = \hat{u}_{obs} = \gamma(1, -\vec{v})$

$E|_o = -\hat{p}\cdot\hat{u}_{obs} = m\gamma.$

An evenly-accelerating particle?

$$\begin{cases} x = a_0^{-1}\cosh(a_0\tau) \\ t = a_0^{-1}\sinh(a_0\tau) \end{cases}$$

$(\hat{e}_0(x))^\mu = (\cosh(a_0\tau), \sinh(a_0\tau), 0, 0)$

$(\hat{e}_1(x))^\mu = (\sinh(a_0\tau), \cosh(a_0\tau), 0, 0)$

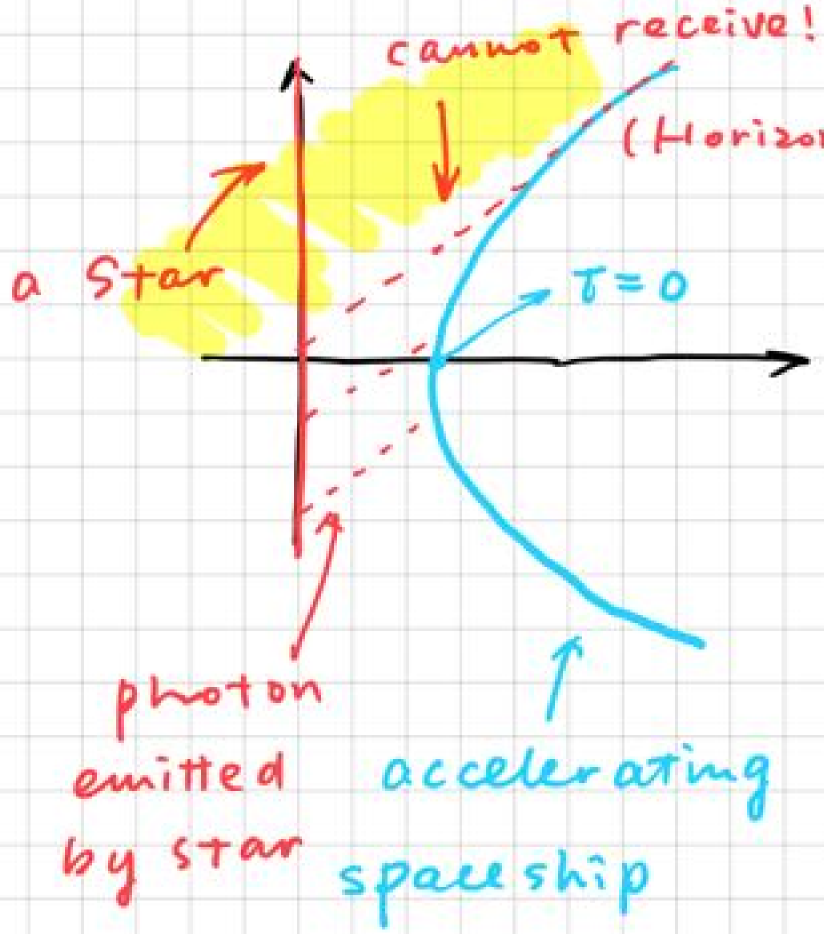
$(\hat{e}_2(x))^\mu = (0, 0, 1, 0)$

$(\hat{e}_3(x))^\mu = (0, 0, 0, 1)$

Star: ω_*

photon.

$$p^\mu = (\hbar\omega_*, \hbar\omega_*, 0, 0)$$



$$E|_0 = -\hat{p}\cdot\hat{u}_0$$

$$= \hbar\omega_*(\cosh(a_0\tau) - \sinh(a_0\tau))$$

$$= \hbar\omega_* e^{-a_0\tau}$$

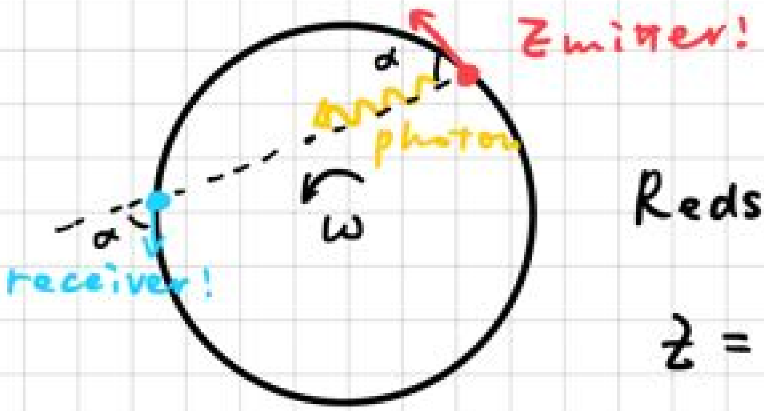
$\tau<0$ opposite-moving blueshift

$\tau>0$ ..... redshift (exponentially)

Rindler observor → Rindler spacetime

· QM? Unruh effect

Einstein's Disk



Q: $\lambda_e \stackrel{?}{=} \lambda_R$

Redshift factor:

$$z = \frac{\lambda_r - \lambda_e}{\lambda_e} = \frac{\lambda_r}{\lambda_e} - 1$$

$$\begin{cases} \text{Emitter: } \hat{u}_e & E_e = -\hat{p}\cdot\hat{u}_e \\ \text{Receiver: } \hat{u}_r & E_R = -\hat{p}\cdot\hat{u}_R \\ \text{photon: } \hat{p} & \end{cases}$$

$$\frac{\lambda_R}{\lambda_e} = \frac{E_e}{E_R} = \frac{-\hat{p}\cdot\hat{u}_e}{-\hat{p}\cdot\hat{u}_R} = \frac{p^0 u_e^0 - \vec{p}\cdot\vec{u_e}}{p^0 u_R^0 - \vec{p}\cdot\vec{u_R}}$$

$$\hat{u} = (\gamma, \gamma\vec{v})$$

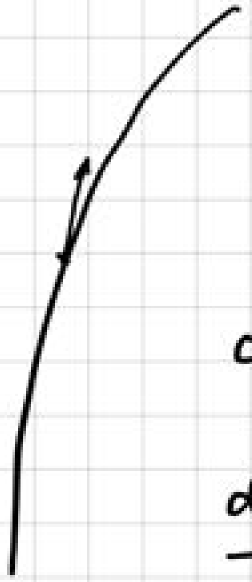
rotating at const v $\rightarrow$ $|\vec{v}| = \omega r$

$$\Rightarrow \frac{\lambda_R}{\lambda_e} = 1$$

Tensor Analysis

1. Vector?

vector ≈ directional derivative.



$x^{\mu}(\lambda)$

$$t^{\mu} = \frac{dx^{\mu}}{d\lambda} \quad \text{(depends on coordinate)}$$

consider an arbitary function f

$$\left.\frac{df(x^{\mu}(\lambda))}{d\lambda}\right|_{\lambda=0} = \left.\frac{df}{dx^{\mu}} \cdot \frac{dx^{\mu}}{d\lambda}\right|_{\lambda=0}$$

$$= \left(\frac{dx^{\mu}}{d\lambda}\right)\frac{d}{dx^{\mu}} f$$

$$\rightarrow \quad \underbrace{\left.\frac{d}{d\lambda}\right|_{\lambda=0}} = \underbrace{\left(\frac{dx^{\mu}}{d\lambda}\right)}\underbrace{\frac{d}{dx^{\mu}}}$$

directional derivative (tangent vector) relative to a world line

t^{μ} ↓ component

$\hat{e}_{\mu}$ → base vector (directional derivative along coordinate axis)

$\left\{\frac{d}{dx^{\mu}}\right\} \rightarrow$ base

$x^{\mu} \to x^{\mu'} = \Lambda^{\mu'}{}_{\nu} x^{\nu}$ (Lorentz rotation)

require the vector indep of coordinates

$$v^{\mu} \frac{\partial}{\partial x^{\mu}} = v^{\mu'} \frac{\partial}{\partial x^{\mu'}}$$

as $v^{\mu'} = \Lambda^{\mu'}{}_{\nu} v^{\nu}$

$\to \hat{e}_{\mu'} = \Lambda^{\nu}{}_{\mu'} \hat{e}_{\nu}$

in electrodynamics.

$v^{\mu} \longrightarrow$ counter-covariant vector

$\hat{e}_{\mu} \longrightarrow$ covariant vector.

Tangent vector $T_p(M)$

Co-vector (Dualvector. 1-form. covariant vector)

$\hat{\omega}$, s.t $\hat{v} \to \mathbb{R}$ $\quad \hat{v} \cdot \hat{\omega} = \hat{\eta}_{\mu\nu} v^{\mu} \omega^{\nu}$

$T^{*}_{p}(M)$

$\hat{\omega}(\hat{v}) \in \mathbb{R}. \to \langle \hat{\omega}, \hat{v} \rangle \in \mathbb{R}.$

$\{\hat{\omega}\}$ vector space.

$\langle a\hat{\omega} + b\hat{\eta}, c\hat{v} + d\hat{w} \rangle = ac \langle \hat{\omega}, \hat{v} \rangle$

$+ \cdots$

basis co-vector?

require $\hat{\theta}^{\mu}(\hat{e_\nu}) = \delta^{\mu}{}_{\nu}$. s.t $\forall\ \hat{\omega} = \omega_\mu \hat{e}^{\mu}$

and $\hat{\omega}(\hat{v}) = \omega_\mu \hat{\theta}^{\mu}(v^\nu \hat{e_\nu})$

$$= \omega_\mu v^\nu \hat{\theta}^{\mu} \hat{e_\nu}$$

$$= \omega_\mu v^\mu$$

Hence:

$$v^{\mu'} = \Lambda^{\mu'}{}_{\nu} v^\nu$$

$$\omega_{\mu'} = \Lambda^{\nu}{}_{\mu'} \omega_\nu$$

$$\hat{\theta}^{\mu'} = \Lambda^{\mu'}{}_{\nu} \theta^\nu$$

example of covariant vector:

gradient of a scalar function.

$df = \nabla f$.

$$df(\hat{v}) = \langle df, \hat{v} \rangle = \frac{df}{d\lambda}\Big|_{\lambda=0} \in R$$

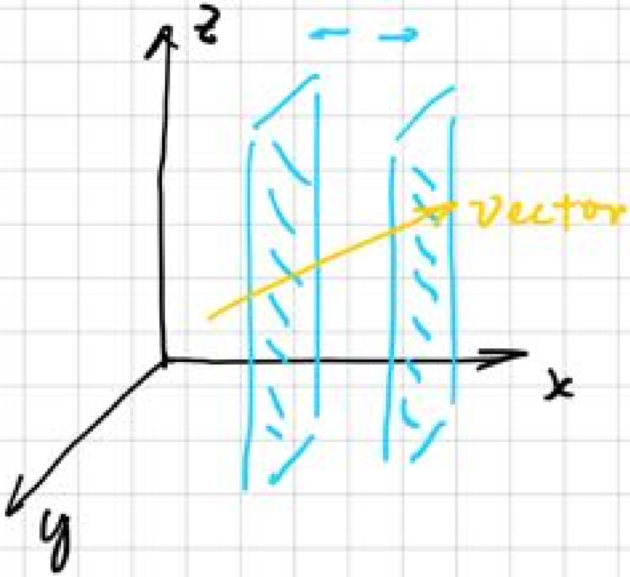
<u>in a specific coordinate</u> $\frac{df}{dx^\mu} v^\mu$

$f = x \quad df = dx \quad \langle df, \hat{e}_\mu \rangle = \langle dx, \frac{\partial}{\partial x} \rangle = 1$

$\Rightarrow \hat{\theta}^\mu = dx^\mu$

$$\therefore \langle \hat{\theta}^\mu, \hat{e}_\nu \rangle = \hat{\theta}^\mu(\hat{e}_\nu)$$
$$= \langle dx^\mu, \frac{\partial}{\partial x^\nu} \rangle$$
$$= \delta^\mu{}_\nu$$

1-form ~ {codimension 1 surfaces}



Hence for a function

$$df = \underbrace{\partial_\mu f}_{\omega_\mu} dx^\mu$$

$$\omega_\mu \quad (\omega_\mu \rightarrow \eta^{\nu\mu}\omega_\mu)$$

Tensor (k,l) - tensor

$$\hat{T} . \quad \underbrace{T_p \otimes \cdots \otimes T_p}_{k} \otimes \underbrace{T_p^* \otimes \cdot T_p^*}_{l}$$

or

$$\hat{T} \{ \underbrace{T_p^* \otimes \cdots \otimes T_p^*}_{k} \otimes \underbrace{T_p \otimes \cdots \otimes T_p}_{l} \} \rightarrow \mathbb{R}$$

example: $(0,0) \rightarrow$ scalar

$(1,0) \rightarrow$ vector

$(0,1) \rightarrow$ co-vector

$(0,2) \rightarrow$ metric tensor

expansion.

$$\hat{T}(k,l) = T^{\mu_1 \cdots \mu_k}{}_{\nu_1 \cdots \nu_l}\, \hat{e}_{\mu_1} \otimes \cdots \otimes \hat{e}_{\mu_k} \otimes \hat{\theta}^{\nu_1} \otimes \cdots \otimes \hat{\theta}^{\nu_l}$$

$$\hat{v} = v^{\mu} \hat{e_{\mu}} \rightarrow (1,0)$$

$$\hat{\omega} = \omega_{\mu} \theta^{\mu} \rightarrow (0,1)$$

$$\hat{g} = g_{\mu\nu} \hat{\theta}^{\mu} \hat{\theta}^{\nu} \rightarrow (0,2)$$

Tensor product

$(k,l), (m,n) \rightarrow (k+m, l+n)$ Tensor.

$$T^{\mu_1 \cdots \mu_k}{}_{\nu_1 \cdots \nu_l} \rightarrow T^{\mu_1' \cdots \mu_k'}{}_{\nu_1' \cdots \nu_k'}$$

$$= T^{\mu_1 \cdots \mu_k}{}_{\nu_1 \cdots \nu_l}\, \Lambda^{\mu_1'}{}_{\mu_1} \cdots \Lambda^{\mu_k'}{}_{\mu_k}\, \Lambda^{\nu_1}{}_{\nu_1'} \cdots \Lambda^{\nu_l}{}_{\nu_l'}$$

Metric tensor (0, 2)

$$\hat{g} = g_{\mu\nu}\hat{\theta}^{\mu}\hat{\theta}^{\nu} \longrightarrow \hat{g}\ \hat{v}\otimes\hat{w} \to \mathbb{R}.$$

$$\hat{g}(\hat{v}, \hat{w}) = g_{\mu\nu}\hat{\theta}^{\mu}\otimes\hat{\theta}^{\nu}(v^{\sigma}\hat{e}_{\sigma}\, w^{\rho}\hat{e}_{\rho})$$

$$\overset{\shortparallel}{\hat{v}\cdot\hat{w}} \qquad = g_{\mu\nu}v^{\mu}w^{\nu}$$

in Minkowski space

$$\hat{g} = \eta_{\mu\nu}\hat{\theta}^{\mu}\hat{\theta}^{\nu}$$

(2, 0)-tensor

$g^{\mu\nu}$: inverse of $g_{\mu\nu}$.

$g^{\mu\nu}$, $g_{\mu\nu}$: symmetric. non-degenerate.

(1, 1) - tensor.

example: kronecker δ-function

$$\hat{v} \xrightarrow{I} \hat{v}$$

EM : strength tensor

A_μ gauge potential

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

$$= \begin{array}{cc} & \begin{array}{cccc} \;0 & \;\;1 & \;\;2 & \;\;3 \end{array} \\ \begin{array}{c} 0 \\ 1 \\ 2 \\ 3 \end{array} & \begin{pmatrix} 0 & -E_1 & -E_2 & -E_3 \\ E_1 & 0 & B_3 & -B_2 \\ E_2 & -B_3 & 0 & B_1 \\ E_3 & B_2 & -B_1 & 0 \end{pmatrix} \end{array}$$

$$F_{\mu\nu} \rightarrow F_{\mu'\nu'} = \Lambda^{\mu}{}_{\mu'} \Lambda^{\nu}{}_{\nu'} F_{\mu\nu}$$

gauge transformation

$$A_\mu \rightarrow A_\mu - \partial_\mu \chi$$ scalar function

$F_{\mu\nu}$ gauge inv.

anti-symmetric $F_{\mu\nu} = - F_{\nu\mu}$

Levi-Civita "Tensor" (symbol)

in a flat space time

$$\varepsilon_{\mu\nu\sigma\rho} = \begin{cases} 1 & (\mu,\nu,\sigma,\rho) \text{ even permutation (0,1,2,3)} \\ -1 & (\mu,\nu,\sigma,\rho) \text{ ... odd ...} \\ 0 & \text{otherwise} \end{cases}$$

$\downarrow$

(0, 4) anti-symmetric tensor

Tensor calculation

1. contraction

$$T^{\mu\nu\sigma}{}_{\nu\rho} \to S^{\mu\sigma}{}_{\rho}$$

$$\delta^{\nu}{}_{\lambda} T^{\mu\lambda\sigma}{}_{\nu\rho} \qquad \#$$

$$\delta^{\rho}{}_{\mu} T^{\mu\lambda\sigma}{}_{\nu\rho} \to (S')^{\lambda\sigma}{}_{\nu}$$

· order of indices.

2. raise & lower the indices

$$\hat{v}^{\mu} \to \hat{v}_{\nu} = v^{\mu} g_{\mu\nu}$$

keep the order

$$T^{\alpha\beta\mu}{}_{\delta} = \eta^{\mu\nu} T^{\alpha\beta}{}_{\nu\delta}$$

$$T_{\alpha}{}^{\beta}{}_{\gamma\delta} = \eta_{\alpha\mu} T^{\mu\beta}{}_{\gamma\delta}$$

Symmetrization & Anti-symmetrization.
( ) [ ]

$$\begin{cases} g_{\mu\nu}.\ g^{\mu\nu} \\ F_{\mu\nu}.\ \varepsilon_{\mu\nu\sigma\rho} \end{cases}$$

$$T^{\mu\nu\sigma}{}_{\rho} \longrightarrow T^{(\mu\nu\sigma)}{}_{\rho} = \frac{1}{3!}\Big(\underbrace{T^{\mu\nu\sigma}{}_{\rho} + \cdots + T^{\sigma\nu\mu}{}_{\rho}}_{\text{all permutation}}\Big)$$

$$\longrightarrow T^{[\mu\nu\sigma]}{}_{\rho} = \frac{1}{3!}\Big(T^{\mu\nu\sigma}{}_{\rho} - T^{\mu\sigma\nu}{}_{\rho} + \cdots\Big)$$

(odd permutation ~ −1
even permutation ~ +)

$$T^{(\mu|\nu\sigma}{}_{|\rho)}$$

Properties:

1. $Y_{\mu\nu} = Y_{(\mu\nu)} + Y_{[\mu\nu]}$

2. $X^{\mu\nu} Y_{(\mu\nu)} = X^{(\mu\nu)} Y_{(\mu\nu)}$

$X^{[\mu\nu]} Y_{(\mu\nu)} = 0.$

3. Trace

$$X_{\mu\nu} \rightarrow \mathrm{Tr}(\hat{X}) = X^{\mu}{}_{\mu} = \underbrace{g^{\mu\nu}}_{\text{symmetric}} X_{\mu\nu} = g^{\mu\nu} X_{(\mu\nu)}$$

$$\eta_{\mu\nu} = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix} \quad tr \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix} = 2$$

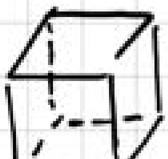
$$\mathrm{Tr}(\eta_{\mu\nu}) = \eta^{\mu\nu} \eta_{\mu\nu} = \delta^{\mu}{}_{\mu} = 4$$ this is right!

Energy – momentum tensor

Fluid $\rightarrow$ continuum

of a large number of particles

"Average" of particle in cell



local parameter

$T(x) \cdot \rho(x)$

element

· viscosity (no $\rightarrow$ perfect fluid) $\left\{ \begin{array}{l} \text{Dust} \\ \text{photon} \\ \text{DM/} \\ \text{DE} \end{array} \right.$

Dust

· no interaction

· comoving frame of the particle

$\rightarrow$ Rest

· move uniformly in other frames.

in a cell (element)

$$N \rightarrow n = \frac{N}{V} \xrightarrow[v_x' = v]{\text{another frame } s'} V' = \frac{V}{\gamma}, \; n' = \gamma n$$

Flux?

$\Delta A = \Delta y \Delta z$

$F_x = \gamma n v$ (4-velocity component?)

Number flux 4-vector

(n. flux) $= n\hat{u}$

note:

$\hat{u}|_{LRF} = (1, 0, 0, 0)$

$\hat{u}|_{\hat{S}} = \gamma(1, v^i)$.

Def: $\hat{N} \equiv n\hat{u}$ (like $\hat{p} = m\hat{u}$)

$$\hat{N} \cdot \hat{N} = -n^2$$

conservation of particle numbers

$$\partial_\mu N^\mu = 0$$

$$\Leftrightarrow \frac{\partial N^0}{\partial t} + \vec{\nabla} \cdot \vec{N} = 0$$

Vector & 1-form
↳ series of hypersurface

consider a special inner product

$$< \hat{v}, dx^{\mu} > = v^{\mu}$$ (ϕ = const, $d\phi$ ~ 1-form)

let $\phi = t$ = const $\quad d\phi = (1, 0, 0, 0)$

$$< \hat{v}, dt > = v^{t} \quad < \hat{p}, dt > = E$$

$$< \hat{N}, dt > = N^{0}$$

in $R^{1,3}$ $\quad t$ = const $\longrightarrow$ series of 3-D space-like hyper-surface.

let $\phi = x$ = const $\longrightarrow (t, y, z)$ (F_x)

$$< \hat{N}, dx > = N^{1}$$

Back to Dust.

energy density

$$\frac{N \cdot m}{V} = nm \xrightarrow{\text{another frame}} (n\gamma)(m\gamma)$$

implies to be a component of a Tensor $\longleftarrow$ $= (nm)\underline{\underline{\gamma^{2}}}$

ρ: (0,0) component of (2,0) Tensor $\hat{T}$

$\rho|_S = nm \rightarrow \rho|_{S'} = nm\gamma^2$

like in Newtonian gravity

$$\nabla^2 \Phi = 4\pi G \underset{\substack{\parallel \\ T_{00} \longrightarrow \hat{T}}}{\rho}$$

$$\hat{T}\,(\text{Dust}) = nm\, \hat{u} \otimes \hat{u}$$

$$= \hat{N} \otimes \underbrace{\hat{p}}_{}$$

4-momentum

$$T^{\alpha\beta} = \underbrace{\hat{T}(dx^\alpha, dx^\beta)}_{\text{(projection to two 1-form)}}$$

flux of this 4-momentum cross surface $x^\beta = \text{const}$

T^{00} Energy density

T^{0i} energy flux
T^{i0} momentum density
} same for dust

T^{ij} stress

$T^{\alpha\beta} \rightarrow$ symmetric

$T^{\alpha\beta} = T^{\beta\alpha}$

T^{ij} : $i=j$ pressure

$i \neq j$ viscosity

Perfect fluid

- choose a comoving frame of fluid element ($p_i = 0$)

$\rightarrow$ for dust, only $T^{00} \neq 0$

- No thermal conduction & viscosity.

$\downarrow$ $T^{0i} = T^{i0} = 0$

$\downarrow$ $T^{ij} = p\delta^{ij}$ (isotropic)

$$T^{\mu\nu}\Big|_{co} = \begin{pmatrix} \rho & & & 0 \\ & p & & \\ & & p & \\ 0 & & & p \end{pmatrix} = (\rho + p)u^\mu u^\nu + p\eta_{\mu\nu}$$

$\rightarrow \hat{T}|_{\text{perfect fluid}} = (p+\rho)\hat{u}\otimes\hat{u} + p\hat{\eta}$

$\rightarrow$ curved space $\hat{\eta} \rightarrow \hat{g}$

$\hat{u}$ is independent of coordinates

$\Rightarrow \hat{T}$ also ...

Eq of State (EOS)

$p = p(\rho)$

- Dust $p = 0$
- Radiation $p = \frac{1}{3}\rho$
- Dark energy (candidate: Cosmological Constant)

$p = -\rho \qquad \hat{T} = -\rho\hat{\eta}$

Conservation law

$\partial_\mu T^{\mu\nu} = 0 \rightarrow \begin{cases} \text{momentum conserv} \\ \text{energy conserv} \end{cases}$

consider perfect fluid

$$T^{\mu\nu} = (\rho + p) u^\mu u^\nu + p \eta^{\mu\nu}$$

$$\rightarrow \partial_\mu \left((\rho + p) u^\mu u^\nu + p \eta^{\mu\nu} \right) = 0$$

$$= \partial_\mu (\rho + p) u^\mu u^\nu + (\rho + p) \partial_\mu (u^\mu u^\nu)$$

$$+ \partial_\mu (p) \eta^{\mu\nu} = 0$$

projection to $\begin{cases} \hat{u}_\parallel \\ \hat{u}_\perp \end{cases}$

$\hat{u}_\parallel$: $\quad u_\nu \partial_\mu T^{\mu\nu} = 0 \quad$ (note $n_\nu n^\nu = -1$)
and $n_\nu \partial_\mu n^\nu = 0$

$$- \partial_\mu (\rho + p) u^\mu - (\rho + p) \partial_\mu u^\mu$$

$$+ \partial_\mu (p) u^\mu = 0$$

$$- \partial_\mu \rho \, u^\mu - (\rho + p) \partial_\mu u^\mu = 0$$

$$\partial_\mu (\rho u^\mu) + p \partial_\mu u^\mu = 0.$$

$\partial_t \rho + \vec{\nabla} (\rho \vec{u}) = 0$

Non-relativistic limit

$$u^i = \gamma (1 \quad v^i) \qquad p \ll \rho \qquad v^i \ll 1$$

$\hat{u}_\perp$: projection operator

$$P^\sigma{}_\nu = \delta^\sigma{}_\nu + u^\sigma u_\nu$$

or

$$P_{\mu\nu} = \eta_{\mu\nu} + u_\mu u_\nu$$

consider $\hat{\omega} \parallel \hat{u}$ ($\omega_\sigma = a u_\sigma$)

$$P^\sigma{}_\nu \omega_\sigma = \omega_\nu - a u_\nu = 0$$

consider $\hat{\omega} \perp \hat{u}$

$$P^\sigma{}_\nu \omega_\sigma = \omega_\nu$$

So $P^\sigma{}_\nu$ can extract the orthogonal component of $\hat{\omega}$

$$P^\sigma{}_\nu \partial_\mu T^{\mu\nu} = 0$$

leave for exercise

$$(p+\rho) u^\mu \partial_\mu u^\sigma + \partial^\sigma p + u^\sigma u^\mu \partial_\mu p = 0 .$$

comoving frame $\sigma = 0$ trivial

$\sigma = i$. $u^i = 0$ $\frac{du^i}{dt} \neq 0$ $\Rightarrow (\rho + p) a^i + \partial^i p = 0$.

$\rightarrow$ non-relativistic $\rho \vec{a} + \nabla p = 0$ Euler Eq.

EM review

$$(\vec{E}^i, \vec{B}^i) \to F_{\mu\nu}\ (F^{\mu\nu})$$

$$F^{0i} = E^i \qquad F^{ij} = \varepsilon^{ijk} B_k$$

Maxwell eqs

$$\to \begin{cases} \partial_\mu F^{\nu\mu} = 4\pi J^\nu \\ \partial^{[\mu} F^{\nu\sigma]} = 0 \end{cases}$$

$$F^{\nu\sigma} = \partial^\nu A^\sigma - \partial^\sigma A^\nu$$ gauge potential

Radiation

note current conservation $\partial_\mu J^\mu = 0$

$$\Rightarrow \partial_\nu \partial^\mu F^{\nu\mu} = 0$$

on the other hand,

$$\partial_\mu \left(\partial^\nu A^\mu - \partial^\mu A^\nu \right) = 4\pi J^\nu$$

use Lorenz gauge $\partial_\mu A^\mu = 0$

$$\to \Box A^\mu = 4\pi J^\nu$$ $\to$ EM radiation

$$A^\nu = c^\nu e^{i\hat{k}\cdot\hat{x}} \qquad \hat{k}\cdot\hat{k} = 0 \text{ (null, travels at c)}$$

$$\hat{c}\cdot\hat{k} = 0 \text{ (transverse wave)}$$

Gravity & SR

Newtonian mechanics: $\frac{d^2\vec{x}}{dt^2} = \vec{a} = \vec{g}$

$$\vec{F} = -G\frac{Mm}{r^2}\vec{e}_r$$

Introduce grav potential

$$\Phi = -G\frac{M}{r}$$

$$\vec{g} = -\nabla\Phi$$

Mass distribution

$$\Phi = -G\sum_i \frac{m_i}{|\vec{x}-\vec{x}_i|}$$

mass (energy) density

$$= -G\int d^3x' \frac{\rho(\vec{x}')}{|\vec{x}-\vec{x}'|}$$

$\nabla^2 \Phi = 4\pi G \rho$ Poisson eq.

instantaneous interaction

not covariant

$\rho \to T^{\mu\nu}$

Given $\rho(x) \to \Phi(x)$

$\nabla^2 \Phi$ still scalar

Multipole Expansion for non-symmetric mass distribution

$$\frac{1}{|\vec{x} - \vec{x}'|} = \frac{1}{r} + \sum_i \frac{x^i x^{i'}}{r^2} + \frac{1}{2} \sum_{ij} (3x^i x^j - r^2 \delta^{ij}) \frac{x^i x^j}{r^3}$$

$D^i = \int x^{i'} \rho(x') d^3 x^{i'}$ → Dipole

$Q^{ij} = \int (\qquad) \rho(\vec{x}') d^3 x'$ → Quadrapole

How to construct a covariant theory of gravity? Einstein, 1907

Inertial frame → special relativity

· Gravity lies everywhere

Einstein: (Global) inertial frame does not exist

only $\exists$ LIF.

originate from Galileo

Weak equivalence principle (WEP)

$m_i = m_g$

$$\begin{cases} \vec{F_i} = m_i \vec{a} \\ \vec{F_g} = m_g \vec{g} \end{cases} \xrightarrow{\text{if } m_i = m_g} \boxed{\vec{a} = \vec{g}}$$

(independent of mass)

Eötvös Torsion $\sim 10^{-9} - 10^{-12}$

Einstein: a free falling object cannot feel its mass

→ LIF

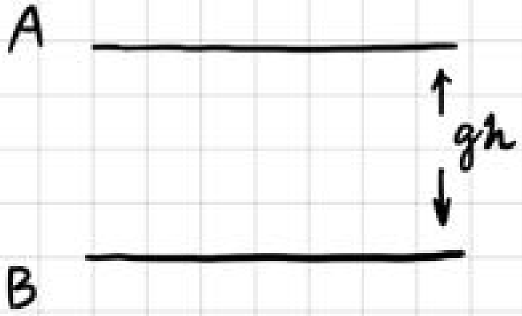
locally, gravity can be simulated by $\vec{a}$

but observer could distinguished them in an non-local condition (relative acceleration from tidal force)

Einstein's Equivalence Principle (EEP)

In a sufficient small region, it's impossible to test the gravity by any experiment. The physical law must be same as in SR.

1911, gravitional redshift



a particle at A with $m_A = E_A$

$$\downarrow \text{ moves to B}$$

$$E_B = m_A + m_A gh$$

$$\downarrow \text{ particle} \to \gamma$$

$$\hbar\omega_B = E_B$$

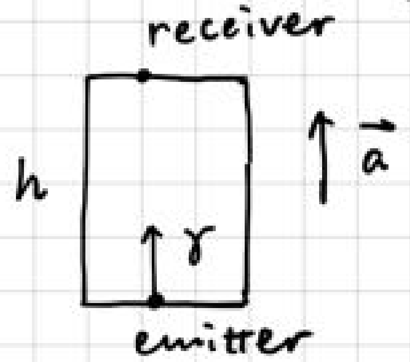
$$\downarrow \text{ goes back to A}$$

$$\hbar\omega_A' = E_A' .$$

conservation of energy.

$$\Rightarrow 1+z = \frac{\omega_B}{\omega_A} = 1+gh \Rightarrow z = gh = \Delta\Phi$$

From another perspective, we use $\vec{a}$ to simulate $\vec{g}$



assumption $v \ll c$ (no SR effect)

h not very large

$$z_E = \frac{1}{2} a t^2 \qquad z_R(t) = h + \frac{1}{2} a t^2$$

$t = 0 \longrightarrow t_1$ 1st photon

$t = \Delta\tau_B \longrightarrow t_1 + \Delta\tau_A$ 2nd photon.

$$\begin{cases} z_R(t_1) - z_E(0) = c t_1 \\ z_R(t_1 + \Delta\tau_A) - z_E(\Delta\tau_B) = c\left(t_1 + \Delta\tau_A - \Delta\tau_B\right) \end{cases}$$

$t_1 \approx \frac{h}{c}$

$$\Rightarrow \quad \Delta\tau_A = \frac{\Delta\tau_B}{1 - gh/c^2}$$

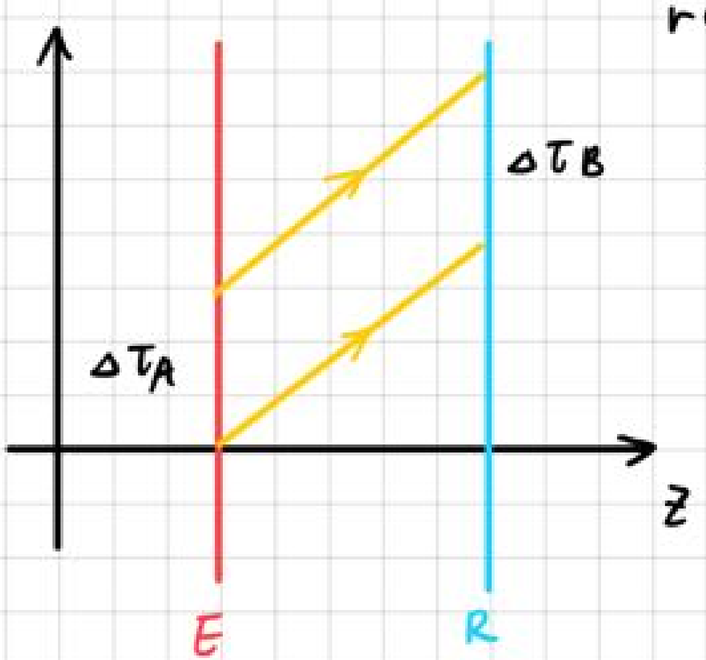
$\downarrow$

redshifted

measured at 1960s.

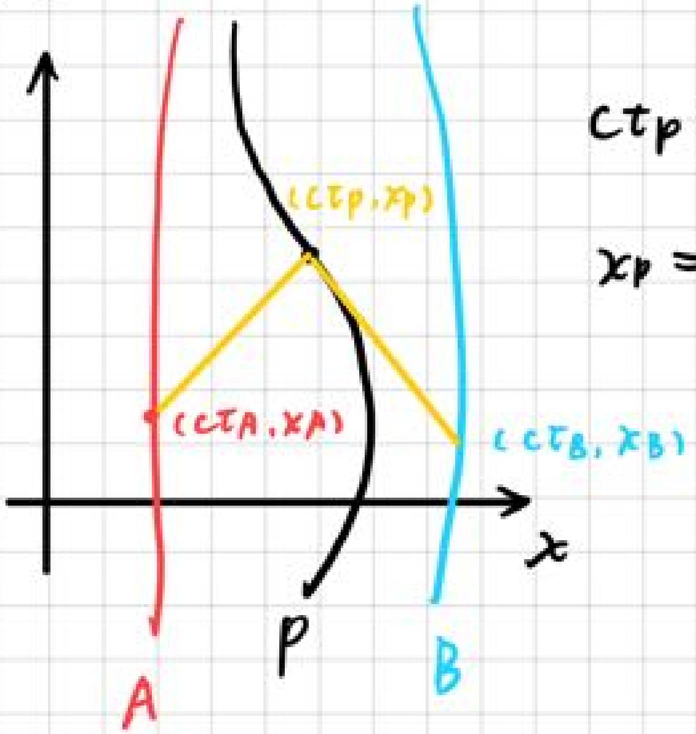
Spacetime no longer flat.

in a Minkowski space-time $\Rightarrow$ no gravitional redshift



Something wrong with $R^{1,3}$

GPS



$$c t_P = \frac{1}{2}\left\{ c(t_A + t_B) + (x_B - x_A) \right\}$$

$$x_P = \frac{1}{2}\left[c(t_B - t_A) + (x_B + x_A) \right]$$

① Time dilations $\leftarrow$ SR.

$v_s \sim 3.9$ km/s

$\beta_s \sim 1.3 \times 10^{-5}$ $\frac{1}{2}\beta_s^2 \sim 0.86 \times 10^{-10}$

② Gravitional redshift

$$\frac{GM_E}{Rc^2} \sim 1.6 \times 10^{-10} s$$

$\downarrow$

error $\sim$ 30 cm.

Manifold

· smooth manifold

i) local flat $\sim R^n$

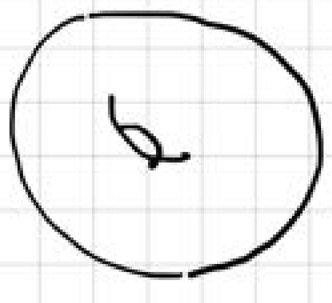
ii) smoothly sewing.

$$R^n \longrightarrow R^{1,n-1}$$

Riemann manifold $\longrightarrow$ Psudo-Riemann manifold

Example:

1. R^n
2. S^n $\quad n=1$ intrinsicly flat
   $\quad n \geq 2$ non-trivial
3. T^n $\quad x^i \sim x^i + L^i$



genus $=1$

4. Riemann surface.

How to describe a manifold?

Embedding

1. curve $x^\mu(\lambda)$

$\rightarrow$ 1-D

2D $x^\mu(\lambda_1, \lambda_2)$.

$\vdots$

nD $x^\mu(\lambda_1, \ldots, \lambda_n)$

2. Hyper-surface co-dim=1

N-dim R^N. $f(x^\mu)=0$

1-constraint $\Rightarrow$ $(n-1)$-dim

more constraints ...

Intrinsic $\begin{cases} \text{locally flat} \\ \text{smoothly paste} \end{cases}$

example: 1) R^2 (x,y) $ds^2 = dx^2 + dy^2$

(r,ϕ) $ds^2 = dr^2 + r^2 d\phi^2$

singularity from coord $(r=0$? ϕ undetermined$)$

2) S^2 (θ, ϕ)

$$ds^2 = d\theta^2 + \sin^2\theta \, d\phi^2$$

$\theta = 0, \pi \longrightarrow$ singularity.

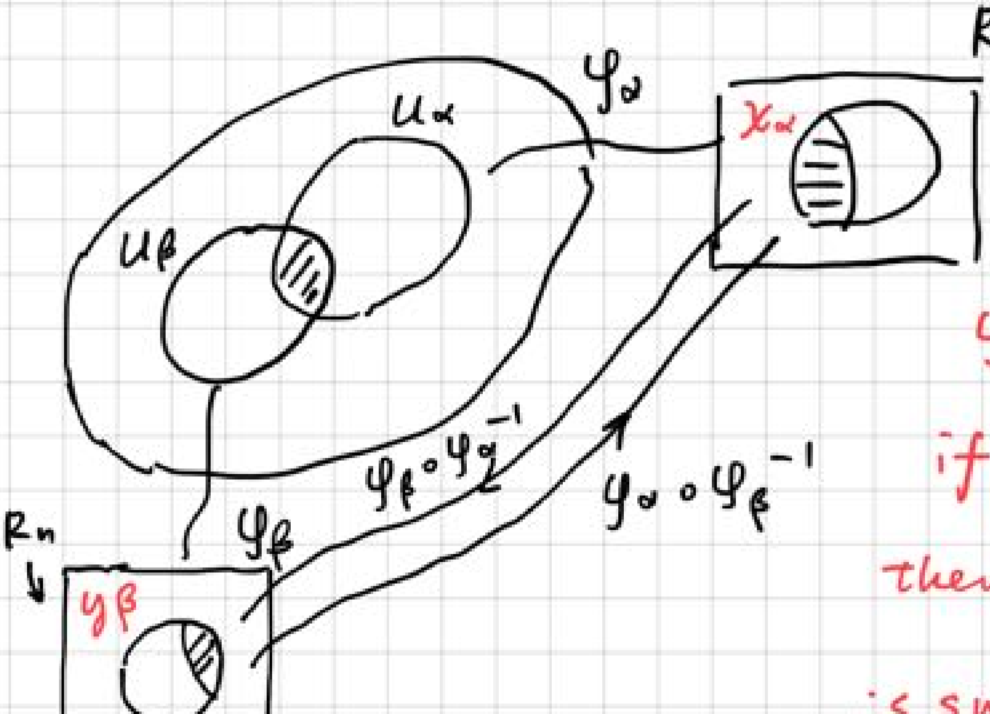
polar projection? $\theta = 0 \rightarrow$ singularity.

S^2 cannot be covered by one coordinate chart.

Manifold

i) $\exists$ topology

ii) a family of coordinate charts $\{(u^i, \varphi^i)\}$.



$\bigcup_i u_i = M$

u_i open set

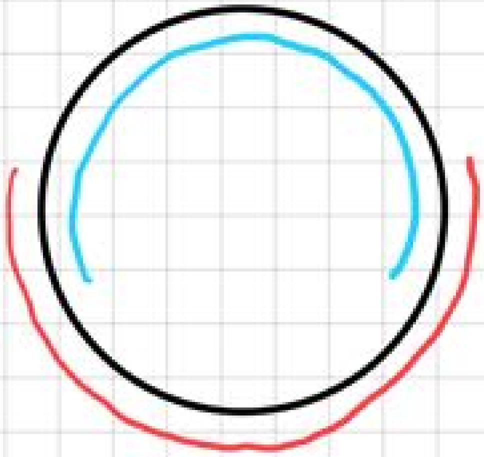
$\varphi_i \; u_i \rightarrow R^n$

if $u_i \cap u_i \neq \phi$

then $\psi_{ij} = \varphi_i \circ \varphi_j^{-1}$

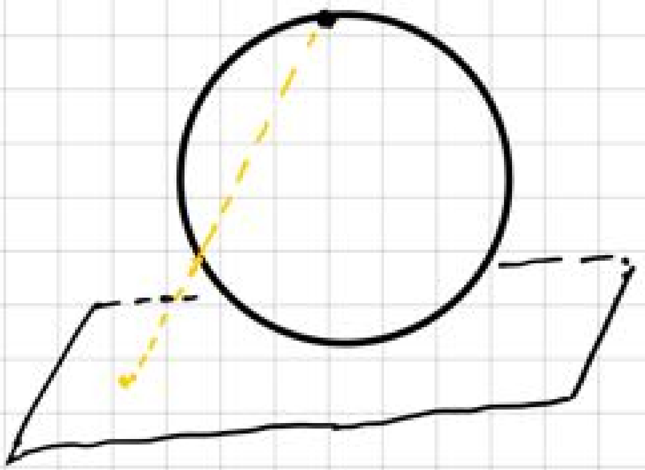
is smooth

example: S^1



R^3. $x_1^2 + x_2^2 + x_3^2 = 1$

U_1 exclude northpole. $x^3 = 1$



$\varphi_1(x_1, x_2, x_3)$

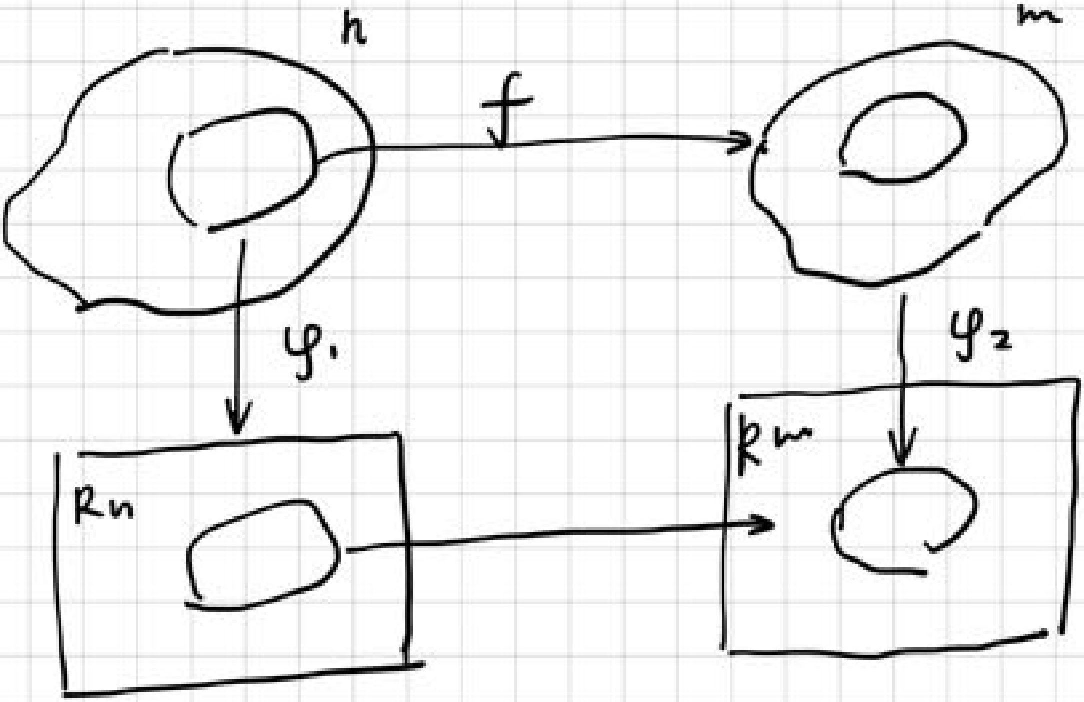
$= (y_1, y_2)$

$= \left(\frac{2x_1}{1-x^3}, \frac{2x_2}{1-x^3}\right)$

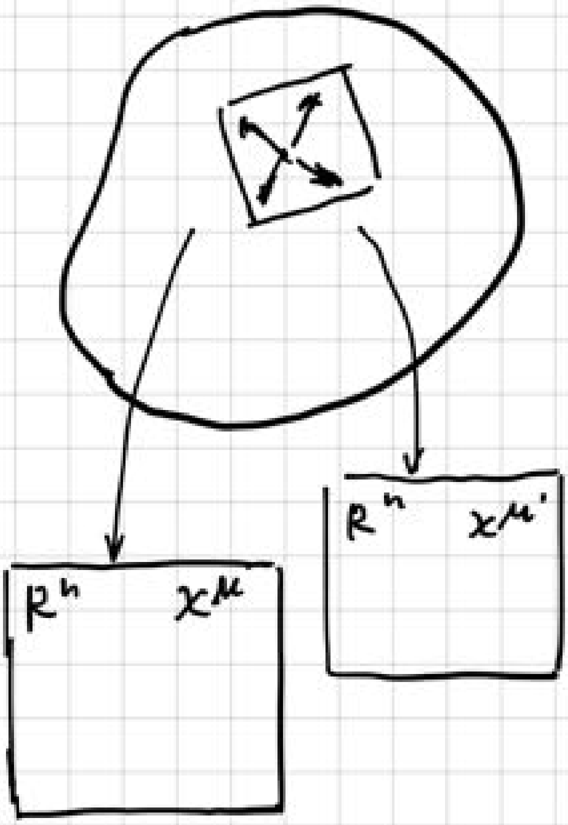
$\varphi_2(x_1, x_2, x_3)$

exclude southpole

...



Tagent space.



$$\hat{x} = x^\mu \hat{e_\mu}$$

$$e_\mu = \frac{\partial}{\partial x^\mu}$$

$$\{\hat{e_\mu}\} \to T_p.$$

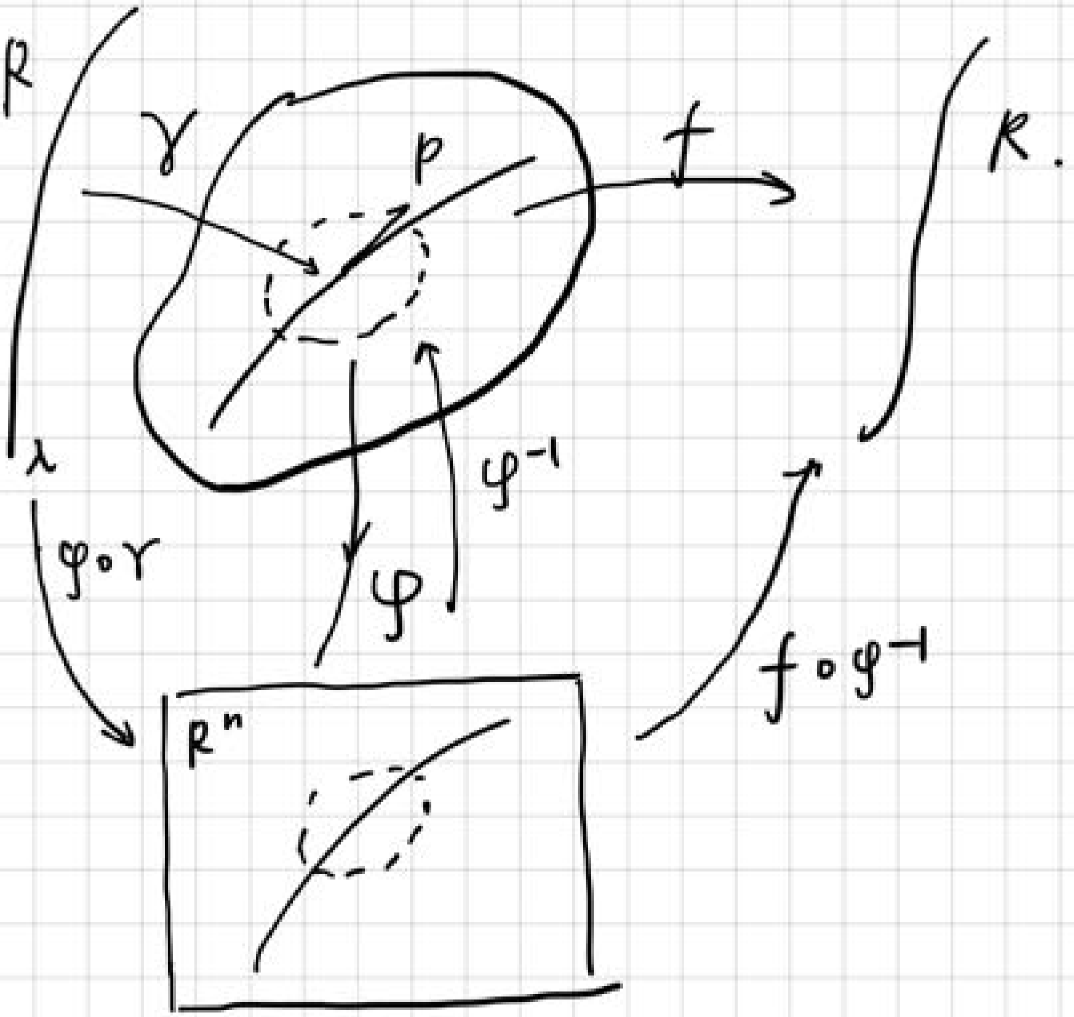
coordinate basis.

$$\hat{v} = v^\mu \hat{e_\mu}$$

$$= v^{\mu'} \hat{e_{\mu'}}$$

$$\Rightarrow \hat{e_{\mu'}} = \boxed{\frac{\partial x^\mu}{\partial x^{\mu'}}} \hat{e_\mu}$$

Differential homomorphism.



$$\dim(T_p(M)) = \dim(M)$$

$$f \in \mathcal{F}(M)$$

$$f \circ \gamma : R \to R$$

$$= (f \circ \varphi^{-1})(\varphi \circ \gamma)$$

$$\frac{d}{d\lambda} f(\lambda)\Big|_\lambda = \frac{d}{d\lambda}\left[(f \circ \varphi^{-1})(\varphi \circ \gamma)\right]$$

$$= \frac{d}{d\lambda}(\varphi \circ \gamma)^\mu \frac{\partial (f \circ \varphi^{-1})}{\partial x^\mu} = \frac{dx^\mu}{d\lambda} \cdot \partial_\mu f$$

Cotangent Space

$\hat{\omega} : \hat{V} \to R.$

$\phi \in \mathcal{F}(M)$ $d\phi \to$ covector.

vector $\to$ depends on location

$\Rightarrow$ vector field $\hat{V}(x)$

$\Leftrightarrow$ co-vector field $\hat{\omega}(x)$

Tensor field $\hat{T}(x)$

$$\hat{T} = T^{\mu_1 \cdots \mu_k}{}_{\nu_1 \cdots \nu_l} \, \partial_{\mu_1} \cdots \partial_{\mu_k} \, dx^{\nu_1} \cdots dx^{\nu_l}$$

$\xrightarrow[\text{another chart}]{\text{transform to}}$ $T^{\mu_1' \cdots \mu_k'}{}_{\nu_1' \cdots \nu_l'}$

$$= \frac{\partial x^{\mu_1'}}{\partial x^{\mu_1}} \cdots \frac{\partial x^{\mu_k'}}{\partial x^{\mu_k}} \frac{\partial x^{\nu_1}}{\partial x^{\nu_1'}} \cdots \frac{\partial x^{\nu_l}}{\partial x^{\nu_l'}} T^{\mu_1 \cdots \mu_k}{}_{\nu_1 \cdots \nu_l}$$

Tensor analysis.

- $\eta_{\mu\nu} \longrightarrow g_{\mu\nu}$

$\partial_\mu \omega_\nu$ tensor?

$\phi \to d\phi$.

$$\partial_{\mu'} \omega_{\nu'} = \frac{\partial x^\mu}{\partial x^{\mu'}} \partial_\mu \left(\frac{\partial x^\nu}{\partial x^{\nu'}} \omega^\nu \right)$$

not a flat space!

$$= \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial x^\nu}{\partial x^{\nu'}} \partial_\mu \omega^\nu + \frac{\partial x^\mu}{\partial x^{\mu'}} \omega_\nu \partial_\mu \left(\frac{\partial x^\nu}{\partial x^{\nu'}} \right)$$

Derivative of a Tensor $\longrightarrow$ no longer a tensor!

Solution:

Derivative $\nearrow$ covariant derivative

Derivative $\searrow$ Antisymmetrization

exterior derivation.

Levi-Civita symbol $\longrightarrow$ Levi-Civita tensor?

Given $\hat{X}, \hat{Y}$ vector fields.

Def: $\underbrace{[\hat{X}, \hat{Y}]}_{\text{Still a vector field.}} f = \hat{X}(\hat{Y}(f)) - \hat{Y}(\hat{X}(f))$

$$[\hat{X}, \hat{Y}] f = \hat{X}\left(Y^\beta \partial_\beta f\right) - \hat{Y}\left(X^\beta \partial_\beta f\right)$$

$$= X^\alpha \partial_\alpha \left(Y^\beta \partial_\beta f\right) - Y^\alpha \partial_\alpha \left(X^\beta \partial_\beta f\right)$$

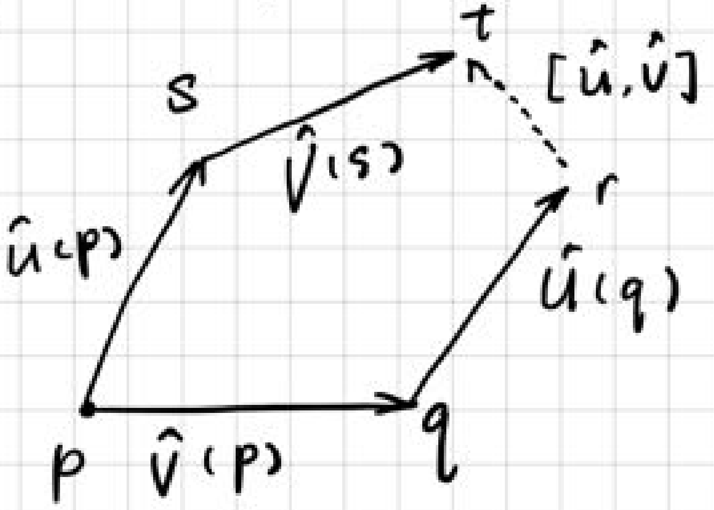
$$= X^\alpha \left(\partial_\alpha Y^\beta\right)\left(\partial_\beta f\right) + X^\alpha Y^\beta \left(\partial_\alpha \partial_\beta f\right)$$

$$- Y^\alpha \left(\partial_\alpha X^\beta\right)\left(\partial_\beta f\right) - Y^\alpha X^\beta \left(\partial_\alpha \partial_\beta f\right)$$

$$= \left[X^\alpha \partial_\alpha Y^\beta - Y^\alpha \partial_\alpha X^\beta\right] \partial_\beta f$$

$$\therefore [\hat{X}, \hat{Y}]^\beta = X^\alpha \partial_\alpha Y^\beta - Y^\alpha \partial_\alpha X^\beta$$

geometry interpretation



$$\vec{rt} = \hat{u}(p) - \hat{u}(q) + \hat{v}(s) - \hat{v}(p).$$

$$= \cdots + \partial_\alpha V_\beta \hat{e_\beta} u^\alpha$$

$$= [\hat{u},\hat{v}].$$

Metric Tensor. $g_{\mu\nu}$

$\left\{ \begin{array}{l} \text{Non-degenerate } (0,2) \text{ tensor} \\ \text{symmetric.} \end{array} \right.$

$\rightarrow$ canonical form

$(\underbrace{-\ -}_{\text{time}},\ \underbrace{+\ +\ +\ \cdots}_{\text{space}})$

· "Distance"

$t+s=n$ signiture $t-s$.

$t=1$ Pseudo-Riemann $\begin{cases} \text{time like} \\ \text{space like} \\ \text{null} \end{cases}$

Causality. lightcone

weak field approximation.

· $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$

↳ Newtonian potential

EEP → LIF

~ locally flat + $g_{\mu\nu}|_p = \eta_{\mu\nu}$

better → $\partial_\sigma g_{\mu\nu}|_p = 0$

Riemann Canonical Coordinate

however $\partial_\sigma \partial_\rho g_{\mu\nu}|_p \neq 0$

given $\forall\, g_{\mu\nu}\ \exists\, x \to x'$, s.t

Pf: $p \to$ zero point

$$\begin{cases} x = 0 \\ x' = 0 \end{cases}$$

$$g_{\mu'\nu'} = g_{\mu\nu} \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial x^\nu}{\partial x^{\nu'}} \quad (1)$$

$$x^\mu|_p = \left(\frac{\partial x^\mu}{\partial x^{\mu'}}\right)\Big|_p x^{\mu'} + \frac{1}{2}\left(\frac{\partial^2 x^\mu}{\partial x^{\mu'2}}\right)\Big|_p x^{\mu'2} + \cdots \quad (2)$$

metric can also be expand

$$g_{\mu\nu} = (g_{\mu\nu})_p + (\partial_\mu g)\left(\frac{\partial x^\mu}{\partial x^{\mu'}}\right) x^{\mu'} + (\partial_\mu g)\left(\frac{\partial^2 x}{\partial x' \partial x'}\right) x'^2$$

$$+ (\partial^2 g)\left(\frac{\partial x}{\partial x'}\right)^2_p + \cdots \quad (3)$$

$$R^\rho_{\ \mu\nu\sigma} \sim \frac{n^2(n^2-1)}{2}$$

plug $(2)(3) \to (1)$

$$LHS = \underbrace{g_{\mu'\nu'}(0)}_{\frac{n(n+1)}{2}} + \underbrace{(\partial_{\sigma'} g_{\mu'\nu'}(0))}_{n \cdot \frac{n(n+1)}{2}} x^{\sigma'} + \underbrace{\partial_{\sigma'}\partial_{\rho'} g_{\mu'\nu'}(0)}_{\frac{n(n+1)}{2} \cdot \frac{n(n+1)}{2}}$$

$$x^{\sigma'} x^{\rho'} + \cdots$$

$$RHS = \underbrace{\left(\frac{\partial x}{\partial x'}\right)\left(\frac{\partial x}{\partial x'}\right)}_{n^2} g + \left[\underbrace{\left(\frac{\partial x}{\partial x'}\right)\left(\frac{\partial^2 x}{\partial x' \partial x'}\right)}_{n \cdot \frac{n+1}{2}} g + \left(\frac{\partial x}{\partial x'}\right)\left(\frac{\partial x}{\partial x'}\right)\partial' g\right] x'$$

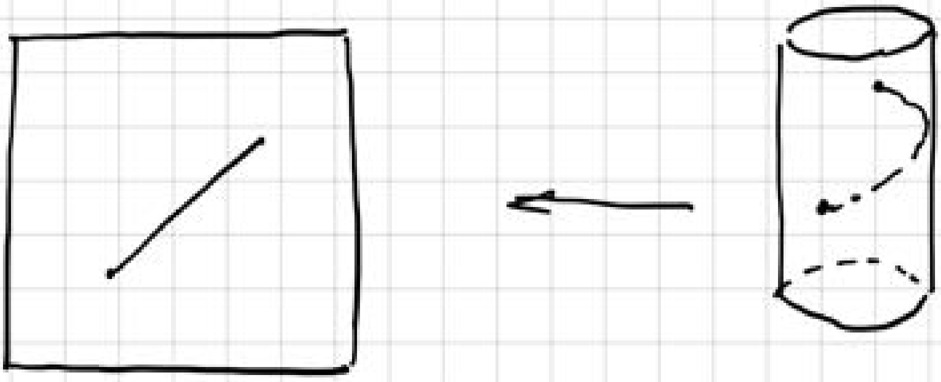
$$+ \left(\underbrace{\frac{\partial}{\partial x'} \frac{\partial^3 x}{\partial x' \partial x' \partial x'}}\, g + \cdots\right) x' x' \qquad \frac{n(n+1)(n+2)}{3!}$$

RNC

$$x^\mu \to x^\mu - \frac{1}{2}\Gamma^\mu_{\sigma\rho}x^\sigma x^\rho$$

$$\Gamma^\mu_{\sigma\rho} = \frac{1}{2}g^{\mu\nu}\left(\partial_\sigma g_{\rho\nu} + \partial_\rho g_{\sigma\nu} - \partial_\nu g_{\sigma\rho}\right)$$

Intrinsic view



$$ds^2 = R^2 d\phi^2 + d\rho^2$$

→ curvature comes from embedding.

$$\Big\downarrow dy = R\,d\phi$$

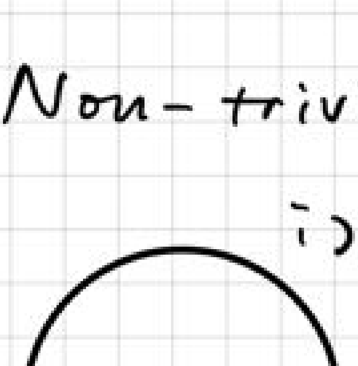
cylinder ⇒ flat. (intrinsic)

$$ds^2 = dy^2 + d\rho^2$$

Non-trivial

induced metric from ↑ 3-D spherical coordinate

i) $ds^2 = R^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right)$



other forms? $x^2 + y^2 + z^2 = a^2$

$$\to z^2 = \sqrt{a^2 - (x^2+y^2)} \to dz = f(x,y)$$

$$\downarrow$$

$$ds^2 = dx^2 + dy^2 + dz^2$$

ii) $\Rightarrow ds^2 = dx^2 + dy^2 - \dfrac{(x\,dx + y\,dy)^2}{a^2 - (x^2+y^2)}$

in such case

$-a^2 = x^2 + y^2$ — coordinate singularity.

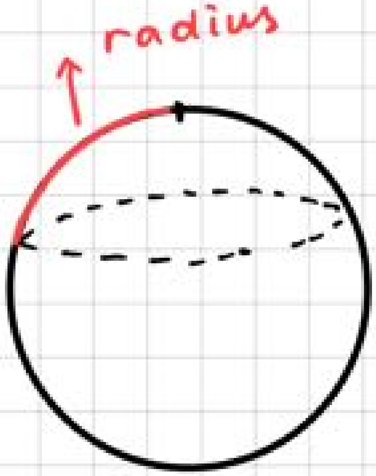
iii) plug in $x = \rho\cos\phi$ $y = \rho\sin\phi$ into ii)

$$ds^2 = \frac{a^2 d\rho^2}{a^2 - \rho^2} + \rho^2 d\phi^2$$

still singularity at $\rho = a$ & $\rho = 0$

from a intrinsic view. radius of a circle

on a sphere.



· length: $L_{AB} = \int_A^B ds$

$$= \int_{\lambda_A}^{\lambda_B} \sqrt{g_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda}}\, d\lambda$$

· Area. let. $g_{\mu\nu} = 0$ if $\mu \neq \nu$.

$ds^2 = g_{11}(dx^1)^2 + g_{22}(dx^2)^2 + \cdots$

sphere

$dA = \sqrt{g_{11} g_{22}}\, d\rho\, d\phi$

$dV = \sqrt{g_{11} g_{22} \cdots g_{NN}}\, dx^1 \cdots dx^N$

consider $ds^2 = \frac{a^2}{a^2-\rho^2} d\rho^2 + \rho^2 d\phi^2$

$$\begin{cases} g_{\rho\rho} = \frac{a^2}{a^2-\rho^2} \\ g_{\phi\phi} = \rho^2 \end{cases}$$

$\rho = R.$ $\quad D = \int_0^R \frac{a}{\sqrt{a^2-\rho^2}} d\rho$

$$= a \arcsin\left(\frac{R}{a}\right)$$

$C = 2\pi R. = 2\pi a \sin\left(\frac{D}{a}\right)$

$$A = \int_0^{2\pi} d\phi \int_0^R d\rho \, \frac{a\rho}{\sqrt{a^2-\rho^2}}$$

$$= 2\pi a^2 \left[1 - \left(1 - \frac{R^2}{a}\right)^{1/2} \right].$$

$$= 2\pi a^2 \left(1 - \cos\left(\frac{D}{a}\right)\right)$$

$D \uparrow$ $C \uparrow$ $A \uparrow$ until $D = \frac{\pi}{2} a$

Difference to Euclid geometry:

$C = 2\pi D$

$A = \pi D^2$

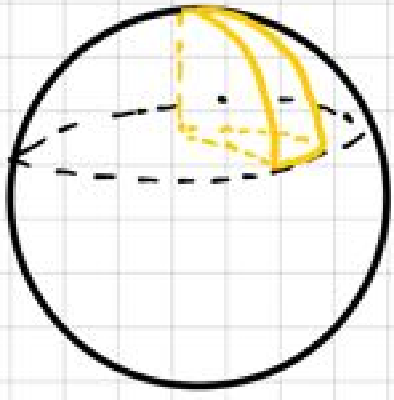
$D \to 0$, locally flat. $\sim R^2$

$$C/D \Big|_{S^2} = \frac{2\pi \sin\left(\frac{D}{a}\right)}{\frac{D}{a}} < 2\pi$$

Expansion of C & A on S^2

$$A = \pi\left[D^2 - \frac{1}{12}kD^2 + \cdots\right]$$

$k = \frac{1}{a^2} \rightarrow$ Gauss curvature



$$k \equiv \frac{3}{\pi} \lim_{D \to 0} \frac{2\pi D - C}{D^3}$$

or

$$\frac{12}{\pi} \lim_{D \to 0} \frac{\pi D^2 - A}{D^4}$$

$$\eta = \theta R = \theta a \sin\left(\frac{D}{a}\right)$$

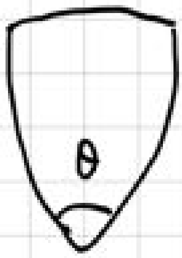
$$= \theta\left(D - \frac{1}{6}kD^3 + \cdots\right)$$

$ds^2 = g_{\mu\nu} dx^\mu dx^\nu$

$$\frac{d^2\eta}{dD^2} = -k\eta$$

Geodesic Deviation Eq.

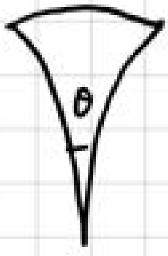
$k > 0$



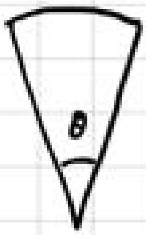
S^2

convex

$k < 0$

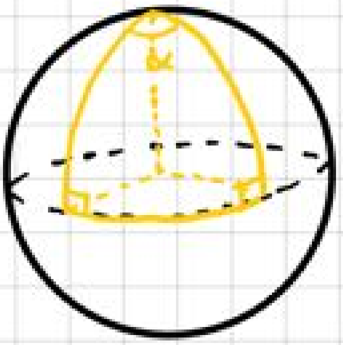


$k = 0$



R^2

Triangle on the sphere



summation of inner angles

$= \pi + \frac{A}{a^2}$

Sub-manifold $x^\mu(u^1 \ldots u^M)$ (M-dims)

Manifold — $g_{\mu\nu}$

$ds^2 = g_{\mu\nu} dx^\mu dx^\nu$

$ds^2 \Big|_{\text{sub manifold}} = g_{\mu\nu} \frac{\partial x^\mu}{\partial u^i} \frac{\partial x^\nu}{\partial u^j} du^i du^j$

$\Rightarrow h_{ij} = g_{\mu\nu} \frac{\partial x^\mu}{\partial u^i} \frac{\partial x^\nu}{\partial u^j}$ induced metric.

volume element

diagonal. $d^N V = \sqrt{g_{11} \cdots g_{NN}}\, dx^1 \ldots dx^N$.

a universal formula

$d^N V = \sqrt{\det(g_{\mu\nu})}\, dx^1 \ldots dx^N$ coordinate independent.

$$x \to x'$$

$$dx^{1'} \cdots dx^{N'} = J\, dx^1 \cdots dx^N$$

$$J = \left| \frac{\partial x'}{\partial x} \right|$$

$$g_{\mu'\nu'} = \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial x^\nu}{\partial x^{\nu'}} g_{\mu\nu}$$

$$\Rightarrow \det(g_{\mu'\nu'}) = J^{-2} \det(g_{\mu\nu})$$

$$\Rightarrow \sqrt{\det(g_{\mu'\nu'})} = J^{-1} \sqrt{\det(g_{\mu\nu})}$$

Hence

$$d^N V = \sqrt{\det(g_{\mu\nu})}\, dx^1 \cdots dx^N$$

$$= \sqrt{\det(g_{\mu'\nu'})}\, dx^{1'} \cdots dx^{N'}$$

$$\int d^N V f \to \text{well defined.}$$

Levi-Civita Tensor (Symbol)

$$\varepsilon_{\mu_1 \cdots \mu_n} = \begin{cases} 1 & \text{even exchange} \\ -1 & \text{odd exchange} \\ 0 & \text{others} \end{cases}$$

Antisymmetric

different coordinates?

$$\tilde{\varepsilon}_{\mu_1' \cdots \mu_n'} |M| = \tilde{\varepsilon}_{\mu_1 \cdots \mu_n} M^{\mu_1}{}_{\mu_1'} \cdots M^{\mu_n}{}_{\mu_n'}$$

let $|M| = \frac{\partial x^{\mu}}{\partial x^{\mu'}}$

$\Rightarrow$

$$\varepsilon_{\mu_1' \cdots \mu_n'} = \left| \frac{\partial x'}{\partial x} \right| \varepsilon_{\mu_1 \cdots \mu_n} \frac{\partial x^{\mu_1}}{\partial x^{\mu_1'}} \cdots \frac{\partial x^{\mu_n}}{\partial x^{\mu_n'}}$$

Tensor density of weight 1

consider $g_{\mu\nu} = \frac{\partial x^{\mu'}}{\partial x^{\mu}} \frac{\partial x^{\nu'}}{\partial x^{\nu}} g_{\mu'\nu'}$

$\Rightarrow |g'| = |J|^{-2} |g|$ (weight -2)

we can combine these two tensor density

$\rightarrow \hat{\varepsilon} = \tilde{\varepsilon} \cdot |g|^{1/2}$

(At an orintable manifold)

Differential p-form

p-form: antisymmetric (0 p) tensor

$\hat{F}$ 2-form $\quad \hat{\varepsilon}$ 4-form $\quad$ df 1-form

n-dim manifold

$$\dim\{p\text{-form}\} = C_n^p = \frac{n!}{p!\,(n-p)!}$$

$\Lambda^p(M)$

Wedge product.

A: p-form

B: q-form

$A \wedge B$: p+q form

$$(A\wedge B)_{\mu_1 \cdots \mu_{p+q}} = \frac{(p+q)!}{p!\,q!} A_{[\mu_1 \cdots \mu_p} B_{\mu_{p+1} \cdots \mu_{q+p}]}$$

example: A, B: 1-form

$$(A\wedge B)_{\mu\nu} = 2A_{[\mu}B_{\nu]} = A_\mu B_\nu - A_\nu B_\mu$$

$$A \wedge B = (-1)^{pq} B \wedge A$$

$\hat{A}$ p-form

$$\hat{A} = A_{\mu_1 \cdots \mu_N} dx^{\mu_1} \otimes \cdots \otimes dx^{\mu_N}$$

$$= \frac{1}{p!} A_{\mu_1 \cdots \mu_N} dx^{\mu_1} \wedge \cdots dx^{\mu_N}.$$

Exterior derivate.

$d \;\; \Lambda^p \rightarrow \Lambda^{p+1}$

$$(d \Lambda^p)_{\mu_1 \cdots \mu_{N+1}} = (p+1)\, \partial_{[\mu_1} A_{\mu_2 \cdots \mu_{N+1}]}$$

$$dW \sim (\underline{\partial_\mu W_\nu - \partial_\nu W_\mu})$$

a well definea tensor. (do not need

covariant derivative)

rilptant $\quad d^2 = 0$

$$d^2 A = \partial_{[\underline{\mu_1}} \partial_{\underline{\mu_2}} \cdots] = 0$$

both symmetric & antisymmetric

$$d(A \wedge B) = dA \wedge B + (-1)^p A \wedge dB.$$

Hodge Dual

n-dim manifold

$$* \Lambda^p \to \Lambda^{N-p}$$

$$(*A)_{\mu_1 \cdots \mu_{N-p}} \equiv \frac{1}{p!} \varepsilon^{\nu_1 \cdots \nu_p}{}_{\mu_1 \cdots \mu_{N-p}} A_{\nu_1 \cdots \nu_p}.$$

note:

$$\varepsilon^{\nu_1 \cdots \nu_p}{}_{\mu_1 \cdots \mu_{N-p}} = g^{\nu_1 \sigma_1} \cdots g^{\nu_p \sigma_p} \varepsilon_{\sigma_1 \cdots \sigma_p \mu_1 \cdots \mu_{N-p}}.$$

$$(**A) = (-1)^{s+p(n-p)} A$$

s: number of negative eigenvalues of metric

$$*(A^{(p)} \wedge B^{(N-p)}) \in \mathbb{R}$$

3D: $*(U \wedge V)_i = \underline{\varepsilon_i{}^{jk} V_j V_k}$ cross product.

Maxwell eqs: F: 2-form

A: 1-form U(1) gauge potential

gauge transformation.

$F = dA \qquad A \to A + d\Lambda$

$\to d(A + d\Lambda)$

$= F.$

✓ $dF = d^2 A = 0$. (Bianchi identity).

$(\partial_{[\mu} F_{\nu\lambda]} = 0)$.

$\partial_\nu F^{\mu\nu} = 4\pi J^\mu$

$\Rightarrow \underbrace{d(*F)}_{\text{3-form}} = 4\pi \underbrace{(*J)}_{\text{3-form}}$

No source. $*J = 0$

$F \longleftrightarrow (*F)$ EM duality.

Volume element

line integral $\int \omega(x) dx$

$\int_\Sigma \omega \rightarrow \mathbb{R}$. ω: n-form

$d^N x = dx^1 \wedge \cdots \wedge dx^N$

$= \frac{1}{n!} \tilde{\varepsilon}_{\mu_1 \cdots \mu_N} dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_N}$

$\longrightarrow \left| \frac{\partial x}{\partial x'} \right| \tilde{\varepsilon}_{\mu_1' \cdots \mu_N'} dx^{\mu_1'} \wedge \cdots \wedge dx^{\mu_N'}$

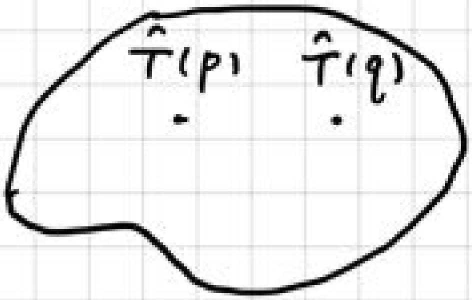
Def: $\sqrt{|g|}\, dx^1 \wedge \cdots dx^n = \hat{\varepsilon}$

$$I = \int \sqrt{|g|}\, d^n x\, f(x) = \int \hat{\varepsilon}\, f(x)$$

Stokes theorem

$$\int_M d^n x \sqrt{|g|}\, \partial_\mu V^\mu = \int_{\partial M} d^{n-1} y \sqrt{|g|}\, n_\mu V^\mu .$$

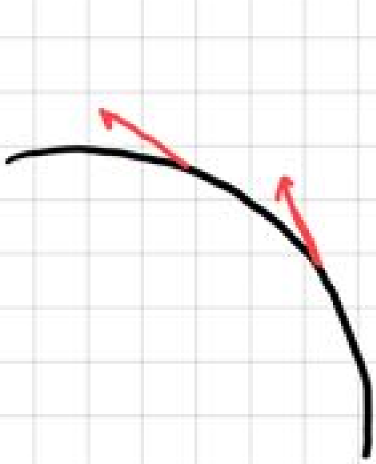
Covariant Derivative



· Derivative have to take account of the information of how p & q are connected (path: tangent vector)

$$X(M) \times \hat{T}(M) \rightarrow \hat{T}(M)$$

$\downarrow$
path
(direction)



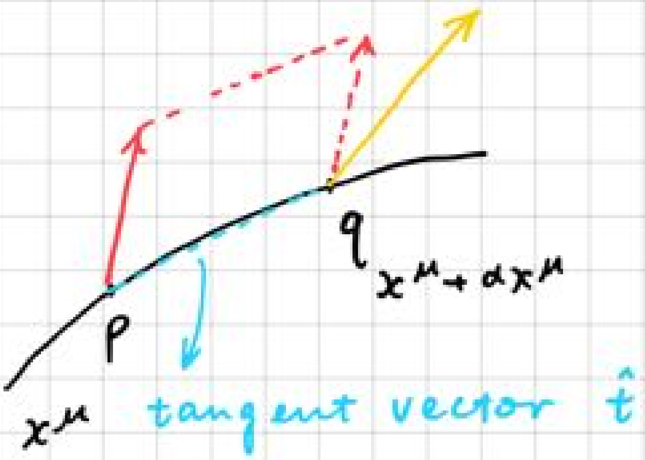
$$\vec{v} = v_r \hat{e}_r + v_\phi \hat{e}_\phi$$

$$\vec{a} = \frac{d\vec{v}}{dt} = a_r \hat{e}_r + v_r \frac{d\hat{e}_r}{dt} + a_\phi \hat{e}_\phi + v_\phi \frac{d\hat{e}_\phi}{dt}$$

· basis vector are changing!

$$\frac{d\hat{e}_r}{dt} = \omega \hat{e}_\phi$$

$$\frac{d\hat{e}_\phi}{dt} = -\omega \hat{e}_r$$



$$\nabla_{\hat{t}} V(x^\mu) = \lim_{\Delta\varepsilon \to 0} \frac{\hat{V}(x^\mu + dx^\mu) - \hat{V}(x^\mu + t^\mu \varepsilon)}{\varepsilon}$$

$$\hat{V} = V^\mu \hat{e_\mu}$$

An affine connection ∇ is a map.

$$\nabla \quad \mathcal{X}(M) \times \mathcal{X}(M) \to \mathcal{X}(M)$$

$$(\hat{X}, \hat{Y}) \to \nabla_{\hat{X}} \hat{Y}$$

satisfying: i) $\nabla_{\hat{X}}(\hat{Y} + \hat{\lambda}) = \nabla_{\hat{X}} \hat{Y} + \nabla_{\hat{X}} \hat{\lambda}$

ii) $\nabla_{\hat{X}+\hat{Y}}(\hat{\lambda}) = \nabla_{\hat{X}}(\hat{\lambda}) + \nabla_{\hat{Y}}(\hat{\lambda})$

iii) $\nabla_{f\hat{X}}(\hat{Y}) = f \nabla_{\hat{X}} \hat{Y}$

iv) $\nabla_{\hat{X}}(f\hat{Y}) = \hat{X}(f)\hat{Y} + f \nabla_{\hat{X}} \hat{Y}$.

Take chart (U, φ) coordinate x^μ

s.t $x^\mu \rightarrow \left\{ \hat{e}_\mu = \frac{\partial}{\partial x^\mu} \right\}$.

$$\nabla_{\hat{e}_\mu} \hat{e}_\nu \equiv \Gamma^\sigma_{\mu\nu} \hat{e}_\sigma$$

↳ describe how base vector change

· based on chart

· flat R^n. $\Gamma^\sigma_{\mu\nu} = 0$.

$$\nabla_{\hat{v}} \hat{\omega} = \nabla_{v^\mu \hat{e}_\mu} \omega^\nu \hat{e}_\nu$$

$$= v^\mu \nabla_{\hat{e}_\mu} \omega^\nu \hat{e}_\nu$$

$$= v^\mu \left[\left(\nabla_{\hat{e}_\mu} \omega^\nu \right) \hat{e}_\nu + \omega^\nu \nabla_{\hat{e}_\mu} \hat{e}_\nu \right]$$

$$= v^\mu \left[\left(\partial_\mu \omega^\nu \right) \hat{e}_\nu + \omega^\nu \Gamma^\sigma_{\mu\nu} \hat{e}_\sigma \right]$$

$$= v^\mu \left(\partial_\mu \omega^\nu + \omega^\nu \Gamma^\sigma_{\mu\nu} \right) \hat{e}_\sigma$$

$$\Rightarrow \left(\nabla_\mu \hat{\omega} \right)^\sigma = \partial_\mu \omega^\nu + \omega^\nu \Gamma^\sigma_{\mu\nu}$$

↓ in short

$\nabla_\mu \omega^\sigma$ or $\frac{D\omega^\sigma}{dx^\mu}$

ex: R^2. polar coordinate:

$$\begin{cases} \hat{e_\rho} = \cos\phi\, \hat{e_x} + \sin\phi\, \hat{e_y} \\ \hat{e_\phi} = -\rho\sin\phi\, \hat{e_x} + \rho\cos\phi\, \hat{e_y} \end{cases}$$

$$\nabla_\rho \hat{e}_\rho = \frac{\partial}{\partial\rho} \hat{e_\rho} = 0$$

$$\nabla_\rho \hat{e_\phi} = \frac{1}{\rho} \hat{e_\phi}$$

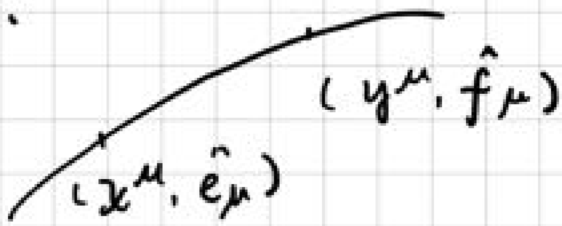
$$\therefore \Gamma^{\mu}_{\rho\rho} = 0 \quad \Gamma^{\phi}_{\rho\phi} = \frac{1}{\rho}$$

$$\nabla_\phi \hat{e_\rho} = \frac{1}{\rho} \hat{e_\phi}$$

$$\nabla_\phi \hat{e_\phi} = -\rho \hat{e_\rho}$$

$$\therefore \Gamma^{\phi}_{\phi\rho} = \frac{1}{\rho} \quad \Gamma^{\rho}_{\phi\phi} = -\rho.$$

$\Gamma^{\sigma}_{\mu\nu}$ is not a tensor!



$$\nabla_{\hat{f_\alpha}} \hat{f_\beta} = \Gamma'^{\sigma}_{\alpha\beta} \hat{f_\alpha}$$

$$\uparrow \quad \hat{f_\alpha} = \frac{\partial x^\mu}{\partial y^\alpha} \hat{e}_\mu$$

$$LHS = \frac{\partial x^\mu}{\partial y^\alpha} \nabla_{\hat{e_\mu}} \left(\frac{\partial x^\nu}{\partial y^\beta} \hat{e_\nu} \right)$$

$$= \frac{\partial x^\mu}{\partial y^\alpha} \left(\frac{\partial}{\partial x^\mu} \left(\frac{\partial x^\nu}{\partial y^\beta} \hat{e_\nu} \right) + \frac{\partial x^\nu}{\partial y^\beta} \Gamma^{\sigma}_{\mu\nu} \hat{e_\sigma} \right)$$

$$= \frac{\partial^2 x^\sigma}{\partial y^\alpha \partial y^\beta} \hat{e}_\sigma + \frac{\partial x^\mu}{\partial y^\alpha} \frac{\partial x^\nu}{\partial y^\beta} \Gamma^\sigma_{\mu\nu} \hat{e}_\sigma$$

$$RHS = \Gamma'^{\sigma}_{\mu\nu} \frac{\partial x^\mu}{\partial y^\sigma} \hat{e}_\mu$$

$$\Rightarrow \Gamma'^{\gamma}_{\mu\nu} = \underbrace{\frac{\partial y^\gamma}{\partial x^\sigma} \frac{\partial^2 x^\sigma}{\partial y^\alpha \partial y^\beta}}_{\text{additional term}} + \underbrace{\frac{\partial y^\gamma}{\partial x^\sigma} \cdot \frac{\partial x^\mu}{\partial y^\alpha} \cdot \frac{\partial x^\nu}{\partial y^\beta} \Gamma^\sigma_{\mu\nu}}_{\text{looks like a tensor}}$$

α, β. symmetric

To tensorfield

linear:

Leibnitz Rule:?

$$\nabla_{\hat{x}} (\hat{T} \otimes \hat{S}) = (\nabla_{\hat{x}} \hat{T}) \otimes \hat{S} + \hat{T} \otimes (\nabla_{\hat{x}} \hat{S})$$

$$\nabla_{\hat{x}} \langle \hat{\omega}, \hat{Y} \rangle = \langle \nabla_x \hat{\omega}, Y \rangle + \langle \hat{\omega}, \nabla_x \hat{Y} \rangle.$$

consider a scalar function.

$$\nabla_{\hat{x}} f = \hat{x}(f) = x^{\mu} \partial_{\mu} f$$

· 1-form? guess

$$\nabla_{\hat{e}_{\mu}} (dx^{\nu}) = \bar{\Gamma}^{\nu}_{\mu\sigma} dx^{\sigma}$$

$$\hat{\omega} = dx^{\nu} \qquad \hat{Y} = \hat{e}_{\sigma}$$

$$\nabla_{\mu} (\delta^{\nu}_{\sigma}) = 0$$

$$\parallel$$

$$\nabla_{\mu} (\langle dx^{\nu}, \hat{e}_{\sigma} \rangle)$$

$$= \langle \nabla_{\mu} dx^{\nu}, \hat{e}_{\sigma} \rangle + \langle dx^{\nu}, \nabla_{\mu} \hat{e}_{\sigma} \rangle$$

$$= \bar{\Gamma}^{\nu}_{\mu\lambda} \langle dx^{\lambda}, \hat{e}_{\sigma} \rangle + \Gamma^{\lambda}_{\mu\sigma} \langle dx^{\nu}, \hat{e}_{\lambda} \rangle$$

$$= \bar{\Gamma}^{\nu}_{\mu\sigma} + \Gamma^{\nu}_{\mu\sigma}$$

$$\therefore \bar{\Gamma}^{\nu}_{\mu\sigma} = - \Gamma^{\nu}_{\mu\sigma}$$

$$\therefore (\nabla_{\mu} \hat{\omega})_{\sigma} = \nabla_{\mu} \omega_{\sigma} = \partial_{\mu} \omega_{\sigma} - \Gamma^{\lambda}_{\mu\sigma} \omega_{\lambda}$$

$$\hat{T}^{(k,l)} = T^{\mu_1 \cdots \mu_k}{}_{\nu_1 \cdots \nu_l}\, \hat{e}_{\mu_1} \otimes \cdots \otimes \hat{e}_{\mu_k} \otimes \hat{dx}^{\nu_1} \cdots \hat{dx}^{\nu_l}$$

$$\begin{aligned}\nabla_\sigma \hat{T} = \partial_\sigma T &+ \Gamma^{\mu_1}_{\sigma\lambda} T^{\lambda \cdots \mu_k}{}_{\nu_1 \cdots \nu_l} \\ &+ \Gamma^{\mu_2}_{\sigma\lambda} T^{\mu_1 \lambda \cdots \mu_k}{}_{\nu_1 \cdots \nu_l} \\ &+ \cdots \\ &- \Gamma^{\lambda}_{\sigma\nu_1} T^{\mu_1 \cdots \mu_k}{}_{\lambda \cdots \nu_l} \\ &\cdots \\ &- \Gamma^{\lambda}_{\sigma\nu_l} T^{\mu_1 \cdots \mu_k}{}_{\nu_1 \cdots \lambda}.\end{aligned}$$

Connection coefficient?

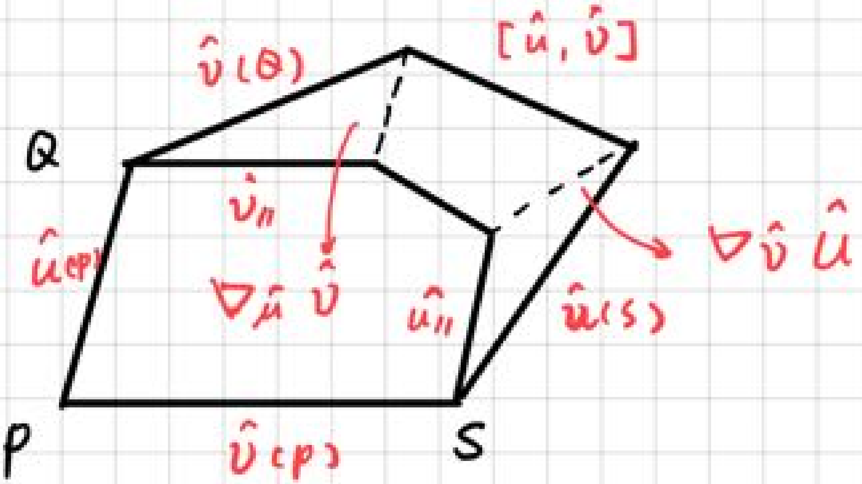
$\Gamma^\sigma_{\mu\nu}$ n-dim $\rightarrow n^3$?

Give $(\Gamma_1)^\sigma_{\mu\nu}$ & $(\Gamma_2)^\sigma_{\mu\nu}$

$(\Gamma_1)^\sigma_{\mu\nu} - (\Gamma_2)^\sigma_{\mu\nu}$: looks like a tensor. (additional term independent on Γ)

$\Gamma^\sigma_{\mu\nu} - \Gamma^\sigma_{\nu\mu}$: looks like a tensor (additional term symmetric to ν, μ).

Torison Tensor



$$\hat{T}(\hat{u},\hat{v})$$

$$= \nabla_{\hat{u}}\hat{v} - \nabla_{\hat{v}}\hat{u} - [\hat{u},\hat{v}]$$

choose $\hat{u} = \hat{e_\mu}$ $\hat{v} = \hat{e_\nu}$, $[\hat{e_\mu},\hat{e_\nu}] = 0$

$$\Rightarrow \hat{T}(\hat{e_\mu},\hat{e_\nu})$$

$$= \nabla_\mu \hat{e_\nu} - \nabla_\nu \hat{e_\mu}$$

$$= \left(\Gamma_{\mu\nu}^{\sigma} - \Gamma_{\nu\mu}^{\sigma}\right)\hat{e_\sigma}$$

Fundamental theorem in Riemann Geometry:

i) Torison less: $\hat{T} = 0 \Rightarrow \Gamma^{\sigma}_{\mu\nu} = \Gamma^{\sigma}_{\nu\mu}$

ii) Metric compatibe:

$$\nabla_\mu g_{\nu\sigma} = 0$$

$$\uparrow$$

$$(\partial_\mu g_{\nu\sigma} = 0)$$

equivalent to:

$$\frac{d}{d\lambda} \langle \underbrace{\hat{X}, \hat{Y}}_{g_{\mu\nu} X^\mu Y^\mu} \rangle = \langle \frac{d}{d\lambda}\hat{X}, \hat{Y} \rangle + \langle \hat{X}, \frac{d}{d\lambda}\hat{Y} \rangle.$$

There is EXACT ONE TORISON -FREE CONNECTION on a given manifold, which is compatible with the same given metric on that manifold.

Proof:

$$\nabla_\rho g_{\mu\nu} = \partial_\rho g_{\mu\nu} - \Gamma^{\lambda}_{\rho\mu} g_{\lambda\nu} - \Gamma^{\lambda}_{\rho\nu} g_{\mu\lambda}$$

$$\nabla_\mu g_{\rho\nu} = \partial_\mu g_{\rho\nu} - \Gamma^{\lambda}_{\mu\rho} g_{\lambda\nu} - \Gamma^{\lambda}_{\mu\nu} g_{\rho\lambda}$$

$$\nabla_\nu g_{\rho\mu} = \partial_\nu g_{\rho\mu} - \Gamma^{\lambda}_{\nu\rho} g_{\lambda\mu} - \Gamma^{\lambda}_{\nu\mu} g_{\rho\lambda}$$

① − (② + ③)

$$\Rightarrow \partial_\rho g_{\mu\nu} - \partial_\mu g_{\rho\nu} - \partial_\nu g_{\rho\mu} + 2\Gamma^{\lambda}_{\mu\nu} g_{\rho\lambda} = 0$$

$$\Rightarrow \Gamma^{\lambda}_{\mu\nu} g_{\rho\lambda} = \frac{1}{2}\left(\partial_\mu g_{\rho\nu} + \partial_\nu g_{\rho\mu} - \partial_\rho g_{\mu\nu}\right)$$

$$\xrightarrow{\times g^{\rho\sigma}} \Gamma^{\sigma}_{\mu\nu} = \frac{1}{2} g^{\rho\sigma}\left(\partial_\mu g_{\rho\nu} + \partial_\nu g_{\rho\mu} - \partial_\rho g_{\mu\nu}\right)$$

Christoffel symbol

* Note: In RNC. $\Gamma^{\sigma}_{\mu\nu} = 0$, but $\partial_\rho \Gamma^{\sigma}_{\mu\nu} \neq 0$. implies that a coordinate can be transformed into RNC.

ex: $ds^2 = dr^2 + r^2 d\phi^2$

$$g_{\mu\nu} = \begin{pmatrix} 1 & \\ & r^2 \end{pmatrix} \qquad g^{\mu\nu} = \begin{pmatrix} 1 & \\ & r^{-2} \end{pmatrix}$$

$$\Gamma^r_{\mu\nu} = \frac{1}{2} g^{rr} \left(\partial_\mu g_{r\nu} + \partial_r g_{r\mu} - \partial_r g_{\mu r} \right)$$

$$\Gamma^r_{rr} =$$

Divergence?

$$\nabla_\mu v^\mu = \frac{1}{\sqrt{|g|}} \partial_\mu \left(\sqrt{|g|}\, v^\mu \right)$$

Pf: $LHS = \partial_\mu v^\mu + \Gamma^\mu_{\mu\sigma} v^\sigma$

$$\Gamma^\mu_{\mu\sigma} = \frac{1}{2} g^{\rho\mu} \left(\partial_\mu g_{\mu\nu} + \partial_\nu g_{\mu\mu} - \partial_\mu g_{\mu\nu} \right)$$

$$= \frac{1}{2} g^{\rho\mu} \partial_\nu g_{\mu\mu}$$

$$RHS. = \frac{1}{\sqrt{|g|}} \left(\partial_\mu \sqrt{|g|}\; v^\mu + \sqrt{|g|}\, \partial_\mu v^\mu \right)$$

$$= \partial_\mu v^\mu + \frac{1}{2} \underbrace{\left(\frac{1}{|g|} \partial_\mu |g| \right)}_{\partial_\mu \ln |g|.} v^\mu.$$

equivalence to $\mathrm{Tr}\left(M^{-1} \otimes \partial_\mu M \right) = \partial_\mu \ln |M|$

$$\delta\left(\ln \mathrm{Det}(M) \right) = \ln \left(\mathrm{Det}(M + \delta M) M^{-1} \right)$$

$$= \ln \mathrm{Det}\left(I + M^{-1} \delta M \right)$$

$$\simeq \ln \mathrm{Det}\left(I + \mathrm{Tr}\left(M^{-1} \delta M \right) \right)$$

$$\simeq 1 + \mathrm{Tr} M^{-1} \delta M$$

Laplace Operator?

$$\nabla^2 \phi = \nabla^\mu \nabla_\mu \phi$$

notice due to the metric compatible condition $\nabla_\mu g_{\sigma\rho} = 0 \to g_{\sigma\rho} \nabla_\mu v^\sigma = \nabla_\mu (g_{\sigma\rho} v^\sigma)$

$$\nabla^2 \phi = \frac{1}{\sqrt{|g|}} \partial_\mu \left(\sqrt{|g|} \partial^\mu \phi \right)$$

Curl?

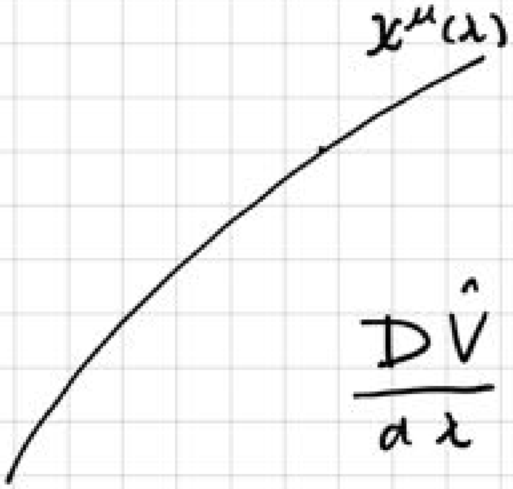
$$(\mathrm{Curl}\ \hat{v})_{\mu\nu}$$

$$= \nabla_\mu v_\nu - \nabla_\nu v_\mu$$

$$= \partial_\mu v_\nu - \partial_\nu v_\mu$$

$$= \frac{1}{2} dV.$$

Instrinsic derivative defined on a curve



e.g $u^\mu(\lambda) = \frac{dx^\mu}{d\tau}$

$$\frac{D\hat{V}}{d\lambda} = \frac{D}{d\lambda}\left(V^\mu \hat{e}_\mu\right)$$

$$= \left(\frac{d}{d\lambda}V^\mu\right)\hat{e_\mu} + V^\mu\left(\frac{D}{d\lambda}\hat{e_\mu}\right)$$

$$\hat{t} = \frac{d}{d\lambda} = t^\nu \frac{d}{dx^\nu} = \left(\frac{dx^\nu}{d\lambda}\right)\frac{d}{dx^\nu}$$

$$\frac{D}{d\lambda}\hat{e_\mu} = \left(\frac{dx^\nu}{d\lambda}\right)\left(\nabla_\nu \hat{e}_\mu\right) = \left(\frac{dx^\nu}{d\lambda}\right)\Gamma^\sigma_{\nu\mu}\hat{e_\sigma}$$

$$\therefore \quad \frac{DV}{d\lambda} = \left(\frac{d}{d\lambda}V^\mu + V^\sigma \Gamma^\mu_{\nu\sigma}\frac{dx^\nu}{d\lambda}\right)\hat{e}_\mu$$

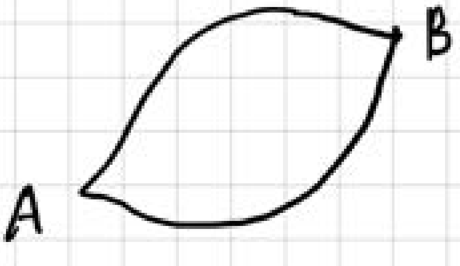
$$\therefore \quad \frac{DV^\mu}{d\lambda} = \left(\frac{dx^\nu}{d\lambda}\right)\left(\partial_\nu V^\mu + \Gamma^\mu_{\nu\sigma}V^\sigma\right)$$

$$= \left(\frac{dx^\nu}{d\lambda}\right)\nabla_\nu V^\mu$$

Parallel Transport

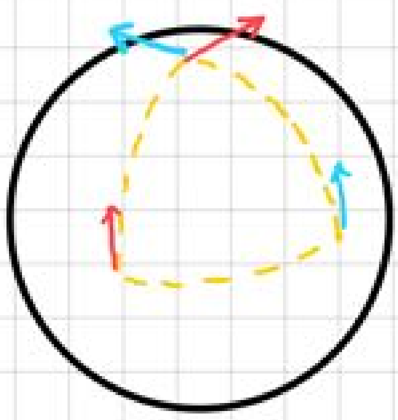
$$\begin{cases} \dfrac{D\hat{T}}{d\lambda} = 0 \\ \hat{T}\big|_{\lambda=0} = \hat{T}_0 \end{cases} \longrightarrow \text{keep the tensor "constant"}$$

at $\mathbb{R}^2$ $\longrightarrow$ path-independent from A to B



(result of 0 curvature)

$S^2 \longrightarrow$ path-dependent



massive particle worldline

$$\hat{u} = \left(\frac{dx^\mu}{d\tau}\right) \hat{e_\mu}$$

$$\hat{a} = \frac{D\hat{u}}{d\tau} = 0$$

4-vector parallel transport in such worldline (geodesic)

$$\sim \frac{d\hat{u}}{d\tau} = 0 \text{ (in flat spacetime)}$$

$$\Updownarrow$$

$$\frac{d^2x^\mu}{d\tau^2} = 0 .$$

$$\frac{d}{d\tau}\left(\frac{dx^\mu}{d\tau}\right) + \frac{dx^\nu}{d\tau}\Gamma^\mu_{\nu\sigma}\frac{dx^\sigma}{d\tau} = 0$$

$$\Rightarrow \frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{\nu\sigma}\frac{dx^\nu}{d\tau}\frac{dx^\sigma}{d\tau} = 0 .$$

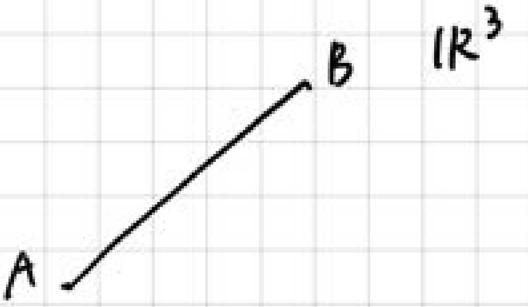
formally:

$$\frac{D\hat{u}}{d\tau} = \nabla_{\hat{u}}\,\hat{u} = 0 .$$

Physically interpretation:

Generalization of Newton's 1st law of motion in curved spacetime

From a math view:



$\longrightarrow$ geodesic: of

extreme "length"

$$\begin{cases} \text{time interval: max} \\ \text{length: min} \end{cases}$$

$$I = \int_{\lambda_A}^{\lambda_B} \sqrt{g_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda}}\, d\lambda$$

$$\delta I = 0$$

$\downarrow$

Euler-Lagrange Eq.

$$\Rightarrow \quad \frac{\partial L}{\partial x^\mu} - \frac{d}{d\lambda}\left(\frac{\partial L}{\partial \left(dx^\mu / d\lambda\right)}\right) = 0$$

plus $2L$

$$\Rightarrow \quad 2L\left(\frac{\partial L}{\partial x^\mu} - \frac{d}{d\lambda}\left(\frac{\partial L}{\partial (dx^\mu/d\lambda)}\right)\right) = 0$$

$$\Rightarrow -\frac{\partial L^2}{\partial x^\mu} + \frac{d}{d\lambda}\left(\frac{\partial L^2}{\partial \dot{x}^\mu}\right) = 2\frac{\partial L}{\partial \dot{x}^\mu}\frac{\partial L}{\partial \lambda}$$

$$LHS = \frac{d}{d\lambda}\left(2 g_{\sigma\mu}\dot{x}^\sigma\right) + \left(\partial_\mu g_{\sigma\rho}\right)\dot{x}^\sigma \dot{x}^\rho$$

$$= 2 g_{\sigma\mu}\ddot{x}^\sigma + 2\dot{x}^\nu\left(\partial_\nu g_{\sigma\mu}\right)\dot{x}^\sigma$$

$$+ (\partial_\mu g_{\sigma\rho})\dot{x}^\rho \dot{x}^\sigma$$

consider

$$\Gamma^\nu{}_{\sigma\rho} = \frac{1}{2} g^{\nu\mu}\left(\partial_\sigma g_{\mu\rho} + \partial_\rho g_{\mu\sigma} - \partial_\mu g_{\sigma\rho}\right)$$

$$\downarrow$$

$$\Gamma_{\nu\sigma\rho} = g_{\nu\mu}\Gamma^\mu{}_{\sigma\rho}$$

$$= 2 g_{\sigma\mu}\ddot{x}^\sigma + 2\dot{x}^\sigma \dot{x}^\rho \Gamma_{\mu\sigma\rho}$$

$$RHS. = \frac{2 g_{\mu\nu}\dot{x}^\nu}{L}\frac{d}{d\lambda}\left(\frac{dI}{d\lambda}\right)$$

$$= 2\left(\frac{\frac{d^2 I}{d\lambda^2}}{\frac{dI}{d\lambda}}\right) g_{\mu\nu}\dot{x}^\nu$$

$$\Rightarrow \ddot{x}^\mu + \Gamma^\mu{}_{\sigma\rho}\dot{x}^\sigma\dot{x}^\rho = \underbrace{2\frac{\ddot{I}}{\dot{I}}\dot{x}^\mu}_{} = 0 \text{ if use affine parameterization } \lambda = aI + b$$

$$\ddot{I} = 0$$

$(L \sim L^2) \xleftarrow{\text{also}}$

$$2K = g_{\mu\nu}\,\dot{x}^{\sigma}\dot{x}^{\nu} = \begin{cases} -1 & \text{timelike} \\ 1 & \text{spacelike} \\ 0 & \text{null} \end{cases}$$

$\delta(2K) = 0 \Rightarrow$ get $\Gamma^{\mu}_{\sigma\rho}$

· charged particle. in external field

$$I = -m\int d\tau + e A_\mu dx^\mu$$

$\delta I = 0.$

geodesic

$\lambda \to \pm\infty$? if meets some (intrinsic) singularity?

$\left\{ \begin{array}{l} \text{Sch BH } r=0 \\ \text{Big Bang } t=0.\ (\text{FRW cosmology}) \end{array} \right.$

↓

Not geodesic completeness

Singularity Theorem

a bunch of geodesic lines

inevitable singularity.

FRW Universe

- cosmological principle

large scale $\begin{cases} \text{homogenous} \\ \text{isotropic} \end{cases}$ → maximally symmetric

Space $\begin{cases} R^n - \text{flat} \\ S^n - \text{closed} \\ H^n - \text{open} \end{cases}$

$$ds^2 = -dt^2 + a^2(t)\left(dx^2 + dy^2 + dz^2\right)$$

$a(t)$ scale factor

$$a(t) = t^q \quad \begin{cases} q = \frac{2}{3} & \text{dust dominated} \\ q = \frac{1}{2} & \text{radiation dominated} \end{cases}$$

$$= e^{Ht} \quad \text{c.c.}$$

$t \to 0$ $a(t) \to 0$ big bang

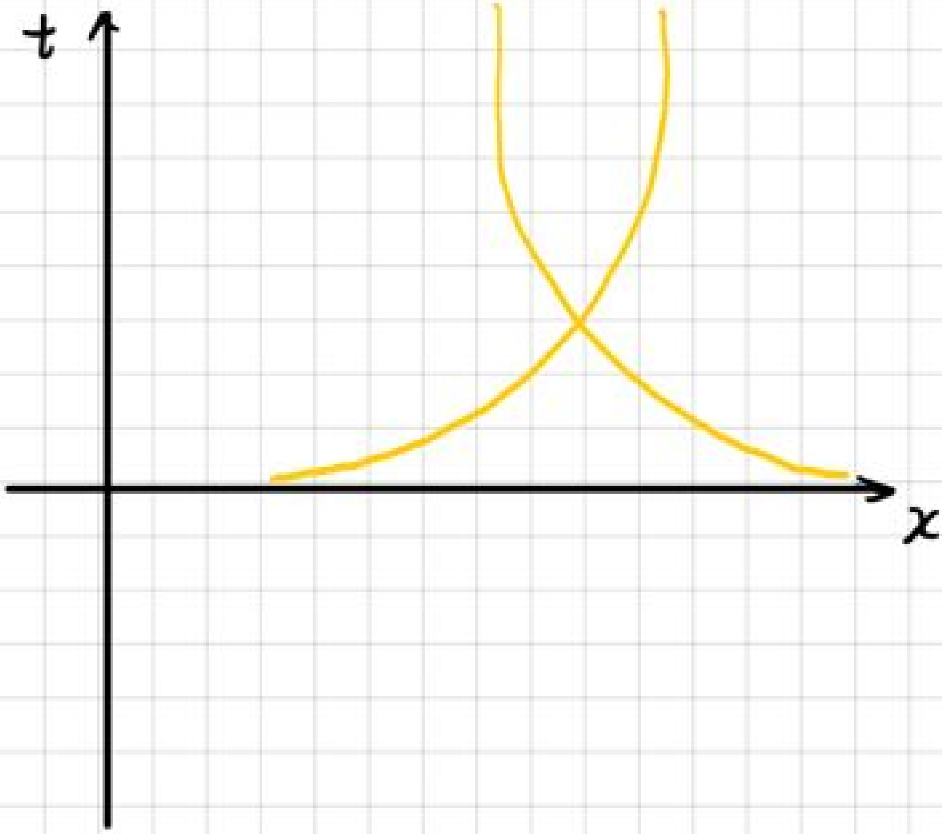
Light cone? let $y = z = 0$

Null path. $d^2 s = 0$

$$x^\mu(\lambda) \to t^\mu = \frac{dx^\mu}{d\lambda}$$

$$\|\hat{t}\| = g_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda} = 0$$

$$-\left(\frac{dt}{d\lambda}\right)^2 + t^{q}\left(\frac{dx}{d\lambda}\right)^2 = 0$$



Null geodesic

$$ds^2 = -dt^2 + a^2(t)\,\delta_{ij}\,dx^i dx^j$$

$$I = \frac{1}{2}\int\left[-\left(\frac{dt}{d\tau}\right)^2 + a^2(t)\,\delta_{ij}\frac{dx^i}{d\tau}\frac{dx^j}{d\tau}\right]d\tau$$

$$x^\mu \to x^\mu + \delta x^\mu$$

$$a(t) \to a(t) + \dot a\,\delta t$$

$$\Rightarrow \delta I = \frac{1}{2}\int\left[-2\left(\frac{dt}{d\tau}\right)\frac{d\delta t}{d\tau} + 2a(t)\,\delta a(t)\,\frac{dx^i}{d\tau}\frac{dx^i}{d\tau}\right]d\tau$$

$$= \int d\tau\left[\left(\frac{d^2t}{d\tau^2}\right)\delta t + a(t)\,\dot a(t)\,\frac{dx^i}{d\tau}\frac{dx^i}{d\tau}\,\delta t\right] = 0$$

$$\Rightarrow \quad \frac{d^2t}{d\tau^2} + a\dot{a}\delta_{ij}\frac{dx^i}{d\tau}\frac{dx^j}{d\tau} = 0$$

$$\Gamma^0_{00} = \Gamma^0_{0i} = 0 \,, \quad \Gamma^0_{ij} = a\dot{a}\delta_{ij}$$

2) $x^i \to x^i + \delta x$

$$\delta I = \frac{1}{2}\int d\tau\left[a^2\, 2\delta_{ij}\frac{dx^i}{d\tau}\frac{d(\delta x^j)}{d\tau}\right] = 0$$

$$= -\int\left[a^2\frac{d^2x^i}{d\tau^2} + 2a\frac{da}{d\tau}\frac{dx^i}{d\tau}\right]\delta x^j\, d\tau$$

$$\Rightarrow \quad \frac{d^2x^i}{d\tau^2} + 2\frac{\dot{a}}{a}\left(\frac{dt}{d\tau}\right)\left(\frac{dx^i}{d\tau}\right) = 0$$

$$\Rightarrow \Gamma^i_{0i} = \frac{\dot{a}}{a} \qquad \Gamma^i_{jk} = 0$$

$$ds^2 = -dt^2 + a^2(t)\left(dx^2 + dy^2 + dz^2\right)$$

$$-\left(\frac{dt}{d\tau}\right)^2 + a^2\left(\frac{dx}{d\lambda}\right)^2 = 0$$

$$\frac{dx}{d\lambda} = \frac{1}{a}\frac{dt}{d\lambda} \xrightarrow{\text{plug}} \text{ in geodesic eq}$$

$$\frac{dt^2}{d\lambda^2} + \frac{\dot{a}}{a}\left(\frac{dt}{d\lambda}\right)^2 = 0 \Rightarrow \frac{dt}{d\lambda} = \frac{\omega_0}{a}.$$

dust dominated: $a \propto t^{3/2}$

$$t^{-1/2} - t_0^{-1/2} = 2w_0(\lambda_0 - \lambda)$$

Finite affine parameter to reach singularity.

Energy - momentum conservation

in flat space:

energy conservation — time translation

momentum . . . — space translation

"Killing symmetry"

$$\partial_\mu T^{\mu\nu} = 0 \longrightarrow \nabla_\mu T^{\mu\nu} = 0$$

$\rightarrow$ $\begin{cases} \text{energy from fluid} \\ \text{energy from gravity field (spacetime BG)} \end{cases}$

$$\partial_\mu T^{\mu\nu} + \Gamma^\mu_{\mu\sigma} T^{\sigma\nu} + \Gamma^\nu_{\mu\sigma} T^{\mu\sigma} = 0$$

$T^{\mu\nu} = (\rho + p) u^\mu u^\nu + p g^{\mu\nu}$. $\rightarrow$ diagonal

$u^\mu = (1\ 0\ 0\ 0)$

in FRW metric

$$T^{\mu\nu} = \begin{pmatrix} \rho & & & \\ & a^{-2}p & & \\ & & a^{-2}p & \\ & & & a^{-2}p \end{pmatrix}$$

$\nu = 0.$

$$\partial_\mu T^{\mu 0} + \Gamma^\mu_{\mu\sigma} T^{\sigma 0} + \Gamma^0_{\mu\sigma} T^{\mu\sigma} = 0.$$

$$\partial_0 T^{00} + \Gamma^\mu_{\mu 0} T^{00} + \Gamma^0_{00} T^{00} + \Gamma^0_{ii} T^{ii} = 0.$$

$$\partial_0 \rho + 3\frac{\dot{a}}{a}\rho + 3a\dot{a}(a^{-2}p) = 0$$

$$\Rightarrow \quad \partial_0 \rho + 3\frac{\dot{a}}{a}(\rho + p) = 0.$$

$\nu = 1$

$$\partial_\mu T^{\mu 1} + \Gamma^\mu_{\mu\sigma} T^{\sigma 1} + \Gamma^1_{\mu\sigma} T^{\mu\sigma} = 0.$$

$$\partial_1 T^{11} + \Gamma^\mu_{\mu 1} T^{11} + 0 = 0$$

$$\Rightarrow \quad \partial_1 p = 0.$$

Introduce EOS. $p = \omega \rho = \begin{cases} 0 & \text{dust} \\ \frac{1}{3} & \text{radiation} \\ -1 & \Lambda \end{cases}$

$$\dot{\rho} + 3\frac{\dot{a}}{a}(1+\omega)\rho = 0$$

$$\Rightarrow \rho \propto a^{-3(1+\omega)}$$

dust $\rho \propto a^{-3}$ (resonable from $\rho \propto V^{-1}$)

radiation $\rho \propto a^{-4}$ (a^{-1} comes from redshift)

c.c. ρ = const.

$\underbrace{\text{Baryon radiation}}_{4\sim5\%}$ c.c $\downarrow$ 68% DM $\downarrow$ 27%

$$E\big|_{t\ \text{fixed}} = \int \rho a^3 (dx\,dy\,dz)$$

Observor in a curved spacetime

Non-coordinate basis

$$\hat{e}_\mu = \frac{\partial}{\partial x^\mu}$$

$$\hat{g}(\hat{e}_\mu, \hat{e}_\nu) = g_{\mu\nu} \quad [\hat{e}_\mu, \hat{e}_\nu] = 0$$

redefine the basis. Let $g_{\mu\nu}$ diagnoal.

$$\hat{e}_m = e^\mu_m \hat{e}_\mu \ , \ \hat{e}_n = e^\nu_n \hat{e}_\nu$$

tetrod

vielbein

$$\hat{g}(e^\mu_m \hat{e}_\mu, e^\nu_n \hat{e}_\nu)$$

$$= e^\mu_m e^\nu_n g_{\mu\nu}$$

Let $e^\mu_m e^\nu_n g_{\mu\nu} = \eta_{mn}$

the flat spacetime → observor's laboratory

$$[\hat{e}_m, \hat{e}_n] = C^p_{mn} \hat{e}_p$$

↳ structure constant

$$\hat\theta^m(\hat e_n) = \delta^m{}_n$$

$$\therefore \hat\theta^m = e^m{}_\mu (dx^\mu)$$

$$e^m{}_\mu e^\mu{}_n = \delta^m{}_n.$$

ex: $ds^2 = dr^2 + r^2 d\theta^2 + r^2\sin^2\theta \, d\phi^2$

$\hat{e_r} \quad \hat{e_\theta} \quad \hat{e_\phi}$

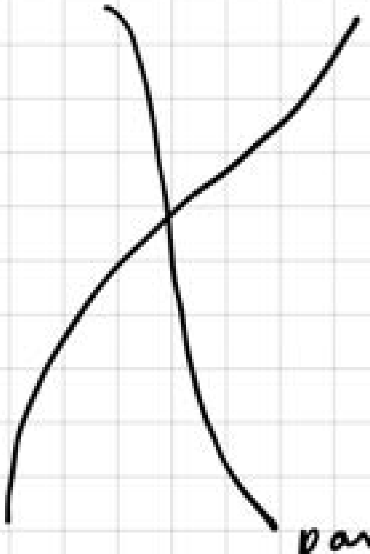
$\hat e_1 = \hat{e_r} \qquad \hat{e_2} = \frac{1}{r}\hat{e_\theta} \qquad \hat{e_3} = \frac{1}{r\sin\theta}\hat{e_\phi}$

$$\hat{e_m}\cdot\hat{e_n} = \delta^m{}_n.$$

$$[\hat e_1, \hat e_2] = \left[\frac{\partial}{\partial r}, \frac{1}{r}\frac{\partial}{\partial\theta}\right] = -\frac{1}{r^2}\frac{\partial}{\partial\theta} = -\frac{1}{r}\hat e_2$$

Observor

$x^\mu(\lambda)\{\hat{e_m}\}\hat{e_0} = \hat u \qquad \hat u \cdot \hat u = -1$



$\hat{e_1}\cdot\hat{e_0} = 0 \qquad \hat{e_i}\cdot\hat{e_j} = \delta_{ij}$

any acceleration. $\hat a$

$\{\hat e_m(\tau=0)\} \to \{\hat e_m(\tau=t)\}$?

particle.

$\nabla_{\hat{u}}\, \hat{e}_a(t) = ?$ $\quad \frac{D}{d\tau}\hat{u} = \hat{a} = \nabla_{\hat{u}}\hat{u} \neq 0$.

$$\nabla_{\hat{u}}\, \hat{e}_a = \frac{D}{d\tau}\hat{e}_a = -\Omega_a{}^b\, \hat{e}_b$$

$$\nabla_{\hat{u}}\, (\hat{e}_a \cdot \hat{e}_b) = 0$$

$$\Rightarrow -\Omega_a{}^c\, \hat{e}_c \cdot \hat{e}_b + \hat{e}_a\left(-\Omega_b{}^c\, \hat{e}_c\right) = 0$$

$$\Rightarrow -\Omega_{ab} - \Omega_{ba} = 0$$

$$\Omega_{ab} = -\Omega_{ba}.$$

interprete as tensor

$$\hat{\Omega} = \Omega^{bc}\left(\hat{e}_b \otimes \hat{e}_c\right)$$

$$\nabla_{\hat{u}}\, \hat{e}_a = \Omega^{bc}\left(\hat{e}_b \otimes \hat{e}_c\right)\hat{e}_a$$

project $\hat{\Omega}$ to $\hat{u}$

$$\Omega^{ab} = v^a u^b - u^a v^b + \omega^{ab}$$

$$\begin{cases} v^a u_a = 0 \\ \omega^{ab} u_a = 0. \end{cases}$$

$$\nabla_{\hat{u}} e_{\hat{a}} = -\Omega^{ab} (e_{\hat{a}} \otimes e_{\hat{b}}) e_{\hat{u}} = \hat{a}$$

$$\hat{\Omega} \cdot \hat{u} = (v^a u^b - v^b u^a + \omega^{ab}) u_a$$

$$= v^b$$

$$\nabla_{\hat{u}} \hat{u} = -\hat{\Omega} \cdot \hat{u} \Rightarrow \hat{v} = -\hat{a}$$

$$= \hat{a}$$

$$\nabla_{\hat{u}} e_{\hat{a}} = -\hat{\Omega} \cdot e_{\hat{a}}$$

$$= -\left(-\hat{a} \otimes \hat{u} + \hat{u} \otimes \hat{a} + \hat{\omega}\right) e_{\hat{a}}$$

$$= -\left(-(\hat{a} \cdot e_{\hat{a}}) \hat{u} + (\hat{u} \cdot e_{\hat{a}}) \hat{a} + \underbrace{\hat{\omega} \cdot e_{\hat{a}}}_{\text{want to be } 0}\right)$$

Assume $\hat{\omega} = 0$ Fermi-Walker transport

how the tetord evolution when observor have acceleration

$$\frac{D_F e_{\hat{a}}}{d\tau} = (\hat{a} \cdot e_{\hat{a}}) \hat{u} - (\hat{u} \cdot e_{\hat{a}}) \hat{a}$$

$a = 0 \quad \dfrac{D_F \hat{u}}{d\tau} = \hat{a}$

$\hat{a} = 0 \quad \dfrac{D e_{\hat{a}}}{d\tau} = 0.$

$\{ e_{\hat{a}}(t) \}$ $E = -\hat{p} \cdot \hat{e}_0$

in FRW universe

Null geodesic: $\frac{dt}{d\lambda} = \frac{\omega_0}{a}$ $\frac{dx}{d\lambda} = \cdot$

$$p^{\mu}_{\text{photon}} = \frac{dx^{\mu}}{d\lambda}$$

consider a comoving observer:

$$u^{\mu} = (1 \quad 0 \quad 0 \quad 0)$$

$$E = -\hat{p}_1 \cdot \hat{u}_0 = \frac{\omega_0}{a}$$

$a = 1$

$E = \omega_0$ ($\hbar = 1$, intrinsic frequency)

Emitter

$\cdot$ $a(t_0)$

$$E_e = \frac{\omega_0}{a_1(t_0)}$$

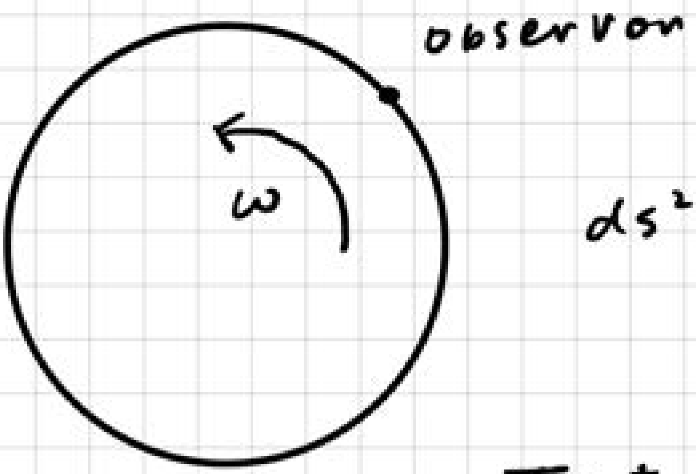
$\downarrow$

Receiver

$a(t_1)$

$$E_r = \frac{\omega_0}{a_1(t_1)}$$

$$\frac{E_r}{E_e} = \frac{a_1(t_0)}{a_1(t_1)}$$



$$ds^2 = -dt^2 + dr^2 + r^2 d\theta^2 + dz^2$$

$$T = t,\ R = r,\ \phi = \theta - \omega t\ \ \chi = z$$

$$g_{MN} = \frac{dx^\mu}{dx^M} \cdot \frac{dx^\nu}{dx^N} g_{\mu\nu}.$$

$$g_{TT} = \frac{dt}{dT}\frac{dt}{dT} g_{tt} + \frac{d\theta}{dT}\frac{d\theta}{dT} g_{\theta\theta}$$

$$= -1 + \omega R^2$$

$$g_{T\phi} = \omega R^2$$

$$ds^2 = -\gamma^{-2} dT^2 + dR^2 + 2R^2 \omega\, dT d\phi + R^2 d\phi^2 + dZ^2$$

$$\gamma = (1 - R^2 \omega^2)^{1/2}$$

$T = \text{const}$

$$\begin{cases} \hat{e}_0 = \gamma \hat{e}_T \quad \hat{e}_1 = \hat{e}_R \\ \hat{e}_2 = \gamma^{-1} R^{-1} \hat{e}_\phi + \gamma R \omega \hat{e}_T \\ \hat{e}_3 = \hat{e}_z \end{cases}$$

$$dl^2 = dR^2 + \gamma^2 R^2 d\phi^2 + dz^2$$

$$dl_\phi = \gamma R d\phi \quad (\text{fixed } R)$$

$$C = 2\pi \gamma R.$$

$$\frac{C}{R} = 2\pi\gamma > 2\pi$$

Proper time

$$d\tau^2 = \gamma^{-2} dT^2$$

Geodesic? in (t, r, θ, z)

$$p^\mu = \left(\frac{dt}{d\lambda}, \frac{dr}{d\lambda}, 0, 0 \right) = (\nu_0, \nu_0, 0, 0).$$

$$\frac{dt}{d\lambda} = \text{const}$$

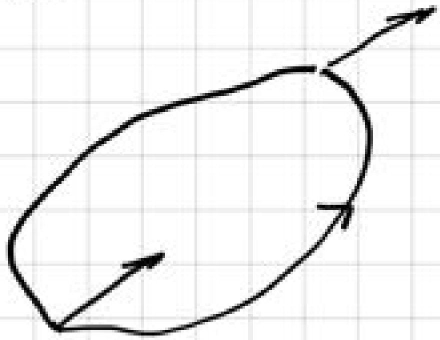
$$p^T = \frac{dT}{dt} p^0 = \nu_0 \quad p^R = \nu_0$$

$$p^\phi = \frac{d\phi}{dt} p^0 = -\omega_0 \nu_0$$

$$\Rightarrow E = -\hat{p}\cdot\hat{u}.$$
$$= -\gamma p^{T} g_{TT} - \gamma p^{\phi} g_{\phi T}$$
$$= \gamma \nu.$$

Curvature

$\mathbb{R}^2$.



path-independent

1) PT. vector. $\longrightarrow$ curvature tensor

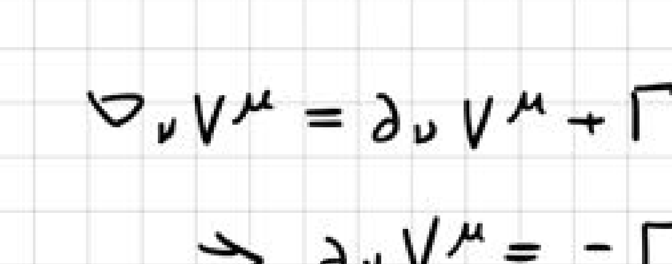
2) $[\partial_\mu, \partial_\nu] = 0 \longrightarrow [\nabla_\mu, \nabla_\nu] \neq 0$

3) geodesic deviation?

$\Downarrow$

Einstein Eq.

local (torsion-free)



$$\nabla_\nu V^\mu = \partial_\nu V^\mu + \Gamma^\mu_{\nu\sigma} V^\sigma = 0$$
$$\Rightarrow \partial_\nu V^\mu = -\Gamma^\mu_{\sigma\nu} V^\sigma$$
$$\frac{V^\mu(q) - V^\mu}{\varepsilon^\nu} = -\Gamma^\mu_{\sigma\nu} V^\sigma .$$

$$\therefore\ V^\mu(Q) = V^\mu - \Gamma^\mu_{\sigma\nu} V^\sigma \varepsilon^\nu$$

$$V^\mu_c(R) = V^\mu(Q) - \Gamma^\mu_{\sigma\nu}(Q) V^\sigma(Q)\,\delta^\nu$$

$$\longrightarrow (\Gamma^\mu_{\sigma\nu} + \partial_\rho \Gamma^\mu_{\sigma\nu}\varepsilon^\rho)$$

$$= V^\mu - \Gamma^\mu_{\sigma\nu} V^\sigma \varepsilon^\nu$$

$$- \Gamma^\mu_{\sigma\nu}(Q)\left(V^\sigma - \Gamma^\sigma_{\lambda\rho} V^\lambda \varepsilon^\rho\right)\delta^\nu$$

cut off at $O(\varepsilon^2)$

$$= V^\mu - V^\kappa \Gamma^\mu_{\nu\kappa}\varepsilon^\nu - V^\kappa \Gamma^\mu_{\kappa\nu}\delta^\nu$$

$$- V^\kappa \left[\partial_\lambda \Gamma^\mu_{\nu\kappa} - \Gamma^\rho_{\lambda\kappa}\Gamma^\mu_{\nu\rho}\right]\varepsilon^\lambda \delta^\nu$$

on the other hand:

$$V_{c'}{}^\mu(R) = V^\mu - \ \ldots$$

$$- V^\kappa \left[\partial_\nu \Gamma^\mu_{\lambda\kappa} - \Gamma^\rho_{\nu\kappa}\Gamma^\mu_{\lambda\rho}\right]\varepsilon^\lambda \delta^\nu$$

Def: $V_{c'}{}^\mu - V_c{}^\mu = V^\kappa \underline{R^\mu{}_{\kappa\lambda\nu}}\, \varepsilon^\lambda \delta^\nu$ Curvature Tensor.

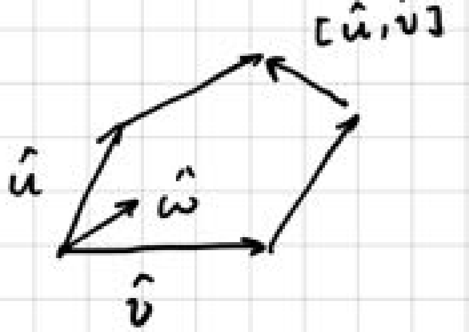
Riemann Tensor

$$\delta V^\mu \rightarrow \hat{\varepsilon}.\hat{\delta}.\hat{V} \quad (1,3)$$

$$R^\mu{}_{\kappa\lambda\nu} = -R^\mu{}_{\kappa\nu\lambda}$$

(path-reversal).

More generalized. (With torsion)



$$\delta\hat{w} = \hat{w}'' - \hat{w}$$

$$\propto W' R_{\cdots} U' V'$$

$$R: \mathcal{X}(M) \times \mathcal{X}(M) \times \mathcal{X}(M) \to \mathcal{X}(M)$$

$$(\hat{w} \quad \hat{u} \quad \hat{v})$$

$$\downarrow$$

$$(\hat{Z}, \hat{X}, \hat{Y})$$

$$R(\hat{X}, \hat{Y}, \hat{Z}) = R(\hat{X}, \hat{Y})\hat{Z}$$

$$= \nabla_{\hat{X}} \nabla_{\hat{Y}} \hat{Z} - \nabla_{\hat{Y}} \nabla_{\hat{X}} \hat{Z}$$

$$- \nabla_{[\hat{X}, \hat{Y}]} \hat{Z}$$

Torsion Tensor

$$\hat{T}(\hat{X}, \hat{Y}) = \nabla_{\hat{X}} \hat{Y} - \nabla_{\hat{Y}} \hat{X} - [\hat{X}, \hat{Y}]$$

$$\hat{T}(\hat{X}, \hat{Y}) = -\hat{T}(\hat{Y}, \hat{X})$$

$$\hat{R}(\hat{Y}, \hat{X}, \hat{Z}) = -\hat{R}(\hat{X}, \hat{Y}, \hat{Z})$$

Multi-linear.

$R(f\hat{X}, g\hat{Y}, h\hat{Z}) = fgh\, R(\hat{X}, \hat{Y}, \hat{Z})$

$f, g, h \in \mathcal{F}(M)$ (non-trivial because

f, g, h are scalar functions).

$\hat{T}(f\hat{X}, g\hat{Y}) = fg\, \hat{T}(\hat{X}, \hat{Y})$.

$\hat{X} = x^{\mu} \hat{e_{\mu}} \longrightarrow \hat{R}(\hat{e_{\mu}}, \hat{e_{\nu}}, \hat{e_{\sigma}})$.

$\hat{e_{\mu'}} = \dfrac{\partial x^{\mu}}{\partial x^{\mu'}} \hat{e_{\mu}}$

$$\hat{T}(f\hat{X}, g\hat{Y}) = \nabla_{f\hat{X}}(g\hat{Y}) - \nabla_{g\hat{Y}}(f\hat{X}) - [f\hat{X}, g\hat{Y}]$$

$$= f(\hat{Y}\nabla_{\hat{X}} g + g\nabla_{\hat{X}}\hat{Y}) - g(\hat{X}\nabla_{\hat{Y}} f + f\nabla_{\hat{Y}}\hat{X}) - [f\hat{X}(g\hat{Y}) - g\hat{Y}(f\hat{X})]$$

$$-\left(fg[\hat{X}, \hat{Y}] + f\hat{Y}\hat{X}(g) - g\hat{X}\hat{Y}(f)\right)$$

$$R(\hat{e_\mu}, \hat{e_\nu}, \hat{e_\sigma}) = \nabla_{\hat{e_\mu}} \nabla_{\hat{e_\nu}} \hat{e_\sigma}$$
$$- \nabla_{\hat{e_\nu}} \nabla_{\hat{e_\mu}} \hat{e_\sigma}$$
$$- \nabla_{[\hat{e_\mu}, \hat{e_\nu}]} \hat{e_\sigma}$$

coordinate basis $[\hat{e_\mu}, \hat{e_\nu}] = 0$

$$= \nabla_\mu \left(\Gamma^\rho_{\nu\sigma} \hat{e_\rho} \right) - \nabla_\nu \left(\Gamma^\rho_{\mu\sigma} \hat{e_\rho} \right)$$

$$= \left(\partial_\mu \Gamma^\lambda_{\nu\sigma} + \Gamma^\rho_{\nu\sigma} \Gamma^\lambda_{\mu\rho} \right) \hat{e_\lambda} - (\mu \leftrightarrow \nu)$$

$$\therefore R^\lambda{}_{\sigma\mu\nu} = \partial_\mu \Gamma^\lambda_{\nu\sigma} + \Gamma^\rho_{\nu\sigma} \Gamma^\lambda{}_{\mu\rho} - (\mu \leftrightarrow \nu)$$

$$\therefore \delta \hat{e_\sigma} = R^\lambda{}_{\sigma\mu\nu} \hat{e_\lambda} (\hat{\delta}^\mu, \hat{\varepsilon}^\nu)$$

Similiarly

$$\hat{T}(\hat{e_\mu}, \hat{e_\nu}) = \nabla_\mu \hat{e_\nu} - \nabla_\nu \hat{e_\mu} - [\hat{e_\mu}, \hat{e_\nu}]$$

$$= \left(\Gamma^{\hat{\lambda}}_{\mu\nu} - \Gamma^{\hat{\lambda}}_{\nu\mu} \right) \hat{e_\lambda}$$

$$\therefore T^\lambda_{\mu\nu} = \Gamma^\lambda{}_{\mu\nu} - \Gamma^\lambda_{\nu\mu}$$

$$g_{\mu\nu} \xrightarrow[\text{metric compatible}]{\text{torsion-free}} \Gamma^{\lambda}_{\mu\nu} \longrightarrow R^{\lambda}{}_{\sigma\mu\nu}$$

$$\text{Flat} \Leftrightarrow R^{\lambda}{}_{\sigma\mu\nu} = 0$$

remember

$$\Gamma^{\lambda}_{\mu\nu} = \frac{1}{2} g^{\lambda\rho} \left(\partial_\mu g \cdots \right)$$

$$R^{\lambda}{}_{\sigma\mu\nu} = \underbrace{\partial\partial g}_{\neq 0 \text{ in curved space.}} + \underbrace{\Gamma\Gamma}_{\text{can be 0 in RNC}} \cdots$$

Symmetry of Riemann Tensor.

$R^{\rho}{}_{\sigma\mu\nu}$ (1, 3)

$\rightarrow R_{\rho\sigma\mu\nu}$ (0, 4)

i) $R_{\rho\sigma\mu\nu} = -R_{\rho\sigma\nu\mu}$

ii) $R_{\rho\sigma\mu\nu} = -R_{\sigma\rho\mu\nu}$

iii) $R_{\rho\sigma\mu\nu} = R_{\mu\nu\rho\sigma}$

iv) $R_{\rho[\sigma\mu\nu]} = 0$ 1st Bianchi identity.

Independent element?

n-dim $\rightarrow n^4$

i) $(\mu, \nu) \rightarrow \frac{n(n-1)}{2} = m$

ii) $(\rho, \sigma) \rightarrow \frac{n(n-1)}{2} = m$

iii) $(\rho\sigma, \mu\nu) \longrightarrow \frac{m(m+1)}{2} = \frac{(n-1)n(n(n-1)+1)}{8}$

iv) non-trivial constraint only if

$(\rho . \sigma . \mu . \nu)$ distinct.

$R_{\rho[\sigma\mu\nu]} = 0$

$$C_n^4 = \frac{n(n-1)(n-2)(n-3)}{24}$$

Independent

$\frac{n^2(n^2-1)}{12}$ RNC. $\partial\partial g_{\mu\nu} \neq 0$.

$n=1$ 0 intrinsic flat

$n=2$ 1

Possible

$$g^{\mu\nu} R_{\rho\sigma\mu\nu} = g^{\rho\sigma} R_{\rho\sigma\mu\nu} = 0$$

only if

$$g^{\rho\mu} R_{\rho\sigma\mu\nu} = R^{\mu}{}_{\sigma\mu\nu}$$
$$= R_{\sigma\nu}$$ ← Ricci Tensor

$R_{\sigma\nu} = R_{\nu\sigma}$ $\quad (R_{\mu\nu\sigma\rho} = R_{\sigma\rho\mu\nu})$

Ricci Scalar (independent of coordinate)

$$R = R^{\nu}{}_{\nu} = g^{\sigma\nu} R_{\sigma\nu}$$

↓ reflect the curved of space

scalar curvature

Einstein Tensor

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu}$$

- symmetric
- $\nabla^{\mu} G_{\mu\nu} = 0$ (to satisfy the conservation of $T^{\mu\nu}$ in EFE)

in RNC.

$$ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu + \frac{1}{12} R_{\mu\nu\sigma\rho} (x^\mu dx^\nu - x^\nu dx^\mu)$$

$$(x^\sigma dx^\rho - x^\rho dx^\sigma)$$

$$R_{\mu\nu\sigma\rho}\big|_{RNC} = \frac{1}{2}\big(\partial_\mu \partial_\sigma g_{\nu\rho} + \partial_\nu \partial_\rho g_{\mu\sigma}$$

$$- \partial_\mu \partial_\rho g_{\sigma\nu} - \partial_\nu \partial_\sigma g_{\mu\rho}\big)$$

(anti-symmetry & symmetry)

2nd Bianchi identity

$$\nabla_{[\lambda} R_{\rho\sigma]\mu\nu} = 0.$$

curvature scalar

$$R_2 = R_{\mu\nu} R^{\mu\nu}$$

$$R_4 = R_{\mu\nu\sigma\rho} R^{\mu\nu\sigma\rho}$$

Pf: $\nabla^\mu G_{\mu\nu} = 0$ consider 2nd Bianchi identity

$$g^{\nu\sigma} g^{\mu\lambda} \big(\nabla_\lambda R_{\rho\sigma\mu\nu} + \nabla_\rho R_{\sigma\lambda\mu\nu} + \nabla_\sigma R_{\lambda\rho\mu\nu}\big)$$

coordinate compatible

$$= \nabla^{\mu} R_{\rho\mu} + \nabla_{\rho}(-R) + \nabla^{\nu} R_{\mu\nu}$$

$$= 2\nabla^{\mu} R_{\rho\mu} - \nabla_{\rho} R = 0.$$

whereas $\nabla^{\mu} G_{\mu\nu}$

$$= \nabla^{\mu}\left(R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu}\right)$$

$$= \frac{1}{2}\left(2\nabla^{\mu} R_{\mu\nu} - \nabla_{\nu} R\right)$$

Weyl Tensor

exclude contraction terms:

$$C_{\mu\nu\sigma\rho} = R_{\mu\nu\sigma\rho} - \frac{2}{n-2}\left(g_{\rho[\mu} R_{\nu]\sigma} - g_{\sigma[\mu} R_{\nu]\rho}\right) + \frac{2}{(n-1)(n-2)} R g_{\rho[\mu} g_{\nu]\sigma}$$

we have

$$g^{\rho\mu} C_{\rho\sigma\mu\nu} = 0$$

in vaccum $R^{\mu\nu}=0 \rightarrow R=0$.

curvature $\longrightarrow C_{\mu\nu\rho\sigma}$.

$$g_{\mu\nu} \rightarrow e^{2n} g_{\mu\nu}$$

$$\Rightarrow C_{\mu\nu\rho\sigma} = \text{const}$$

$$g_{\mu\nu} \longrightarrow \Gamma^{\sigma}_{\mu\nu} \longrightarrow R^{\rho}{}_{\sigma\mu\nu} \longrightarrow R_{\mu\nu} \longrightarrow R$$

Intrinsic

$n=1$ flat

$n=2$ 1 independent element

$$R_{\mu\nu\sigma\rho} = \frac{1}{2} R (g_{\mu\sigma} g_{\nu\rho} - g_{\mu\rho} g_{\sigma\nu})$$

$$\rightarrow R_{\nu\rho} = g^{\mu\sigma} R_{\mu\nu\sigma\rho}$$

$$= \frac{1}{2} R \left(2 g_{\nu\rho} - \delta^{\sigma}{}_{\rho} g_{\sigma\nu} \right)$$

$$= \frac{1}{2} R g_{\nu\rho}$$

$\Rightarrow \quad G_{\nu\rho} \equiv 0$ (no dynamics)

$n=3$ $R_{\mu\nu\sigma\rho}$ - 6 dof

$R_{\mu\nu}$ - 6 dof

$n=2.\ S^2.\quad ds^2 = a^2(d\theta^2 + \sin^2\theta\, d\phi^2)$

$\Gamma^\theta_{\phi\phi} = -\sin\theta\cos\theta$

$\Gamma^\phi_{\theta\phi} = \cot\theta$

$R^\theta{}_{\phi\theta\phi} = \partial_\theta\Gamma^\theta_{\phi\phi} + \Gamma^\theta_{\theta\lambda}\Gamma^\lambda_{\phi\phi} - \partial_\phi\Gamma^\theta_{\phi\theta}$

$+\Gamma^\theta_{\phi\lambda}\Gamma^\lambda_{\theta\phi}$

$= \sin^2\theta$

$\begin{cases} R_{\phi\phi} = \sin^2\theta \\ R_{\theta\theta} = 1 \end{cases} \Rightarrow R = a^{-2} + a^{-2} = \frac{2}{a^2}$

$n=2$ Poincare half plane (hyperbolic)

$ds^2 = \frac{a^2}{y^2}(dx^2 + dy^2)\ ,\ (y>0)$

$\Gamma^x_{xy} = -y^{-1}$

$\Gamma^y_{xx} = y^{-1}$

$\Gamma^y_{yy} = -y^{-1}$

geodesic:

$\begin{cases} \text{i) } x = \text{const} \quad y = e^{\lambda/a} \\ \text{ii) } (x-x_0)^2 + y^2 = r_0^2 \end{cases}$

$$R^{x}{}_{yxy} = y^{-2}$$

$$R_{yy} = -y^{-2}$$

$$R_{xx} = -y^{-2}$$

$R = -\frac{2}{a^2}$ (constant curvature space)

Symmetry of Spacetime?

killing symmetry.

isometry transportation.

$x^{\mu} \to x^{\mu'}$ $\quad g'_{\mu\nu}(x') = g_{\mu\nu}(x)$

$\Rightarrow$ Isometry group $\{x^{\mu} \to x^{\mu'}\}$.

generator?

$$x'^{\alpha} = x^{\alpha} + \varepsilon \xi^{\alpha}$$

$$g'_{\alpha\beta} = \frac{\partial x^{\mu}}{\partial x^{\alpha'}} \frac{\partial x^{\nu}}{\partial x^{\beta'}} g_{\mu\nu}$$

$$\frac{\partial x'^{\alpha}}{\partial x^{\mu}} = \delta^{\alpha}{}_{\mu} + \varepsilon \partial_{\mu} \xi^{\alpha}$$

$$L.H.S = \frac{\partial x'^{\alpha}}{\partial x^{\mu}} \frac{\partial x'^{\beta}}{\partial x^{\nu}} g'_{\alpha\beta}$$

$$= (\delta^{\alpha}{}_{\mu} + \varepsilon \partial_\mu \xi^{\alpha})(\delta^{\beta}{}_{\nu} + \varepsilon \partial_\nu \xi^{\beta})$$

$$(g_{\alpha\beta} + \partial_\lambda g_{\alpha\beta} \varepsilon \xi^{\lambda} + \dots)$$

1^{st} order term $= 0$

$$(\partial_\mu \xi^{\alpha}) \delta^{\beta}{}_{\nu} g_{\alpha\beta} + (\partial_\nu \xi^{\beta}) \delta^{\alpha}{}_{\beta} g_{\alpha\beta}$$

$$+ \delta^{\alpha}{}_{\mu} \delta^{\beta}{}_{\nu} (\partial_\lambda g_{\alpha\beta}) \xi^{\lambda} = 0$$

$$= g_{\alpha\nu}(\partial_\mu \xi^{\alpha}) + g_{\mu\beta}(\partial_\nu \xi^{\beta}) + (\partial_\lambda g_{\mu\nu}) \xi^{\lambda}$$

$$= \partial_\mu (g_{\alpha\nu} \xi^{\alpha}) - \xi^{\alpha} \partial_\mu (g_{\alpha\nu})$$

$$+ \partial_\nu (g_{\mu\beta} \xi^{\beta}) - \xi^{\beta} \partial_\nu (g_{\mu\beta}) + (\partial_\lambda g_{\mu\nu}) \xi^{\lambda}$$

$$= \partial_\mu \xi_\nu + \partial_\nu \xi_\mu - 2 \xi_\gamma \Gamma^{\gamma}_{\mu\nu}$$

$$= \nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu = \nabla_{(\mu} \xi_{\nu)} = 0$$

Killing Eq.

$\nabla_{(\mu} \xi_{\nu)} = 0$ $\quad \hat{\xi}$ killing vector field if $\partial_i g_{\mu\nu} = 0 \Rightarrow \{\partial_i\}$ killing

$\xi^\mu(\lambda) \quad x \to \tilde{x} \quad \tilde{\xi}^\mu = \begin{cases} \bar{\xi}^1 = \text{const} \\ \bar{\xi}^i = 0 \quad (i \neq 1) \end{cases}$

$\Longleftrightarrow \dfrac{\partial \tilde{g}_{\mu\nu}}{\partial \tilde{x}^i} = 0 \quad , \quad \hat{\xi} = \bar{\xi}^1 \partial_{\bar{x}^1}$

$\mathbb{R}^3 \; ds^2 = dx^2 + dy^2 + dz^2.$

$\Rightarrow \{\partial x, \partial y, \partial z\}$

$= d\rho^2 + \rho^2 d\theta^2 + \rho^2 \sin^2\theta \, d\phi^2$

$\to \partial\phi = x\partial y - y\partial x.$

in total 6

$R^{(1,3)} \begin{cases} \partial t \text{ timelike} \\ \partial x \text{ spacelike} \\ (t, x_\mu) \text{ boost} \\ x_\mu, x_\nu \text{ rotation.} \end{cases}$

$[\hat{\xi}_1, \hat{\xi}_2]$ is also a Killing vector field.

$\Rightarrow$ Lie algebra $\rightarrow$ Isometry group.

ex: massive particle

$$p^\mu = m\frac{du^\mu}{d\tau} = ma^\mu$$

Geodesic $\nabla_{\hat{u}}\hat{u} = 0 \quad (\hat{a} = 0)$

$$u^\lambda \nabla_\lambda p_\mu = 0$$

$$\Rightarrow \quad u^\lambda\left(\partial_\lambda p_\mu - \Gamma^\sigma_{\mu\lambda} p_\sigma\right)$$

$$\parallel$$

$$m\frac{dp^\mu}{d\tau}$$

$$u^\lambda \Gamma^\sigma_{\mu\lambda} u_\sigma$$

$$= \frac{1}{2}\left(\partial_\lambda g_{\rho\mu} + \partial_\mu g_{\rho\lambda} - \partial_\rho g_{\lambda\mu}\right) u^\lambda u^\rho$$

$$= \frac{1}{2}\partial_\mu g_{\rho\nu}$$

if $\partial_\mu g_{\rho\nu} = 0. \quad \Rightarrow \quad \frac{dp_\mu}{d\tau} = 0.$

$\hat{k}$ killing vector

$$\frac{d}{d\tau}(\hat{k}\cdot\hat{p}) = 0 \, ?$$

$$\Rightarrow \left(\frac{D\hat{k}}{d\tau}\right)\hat{p} + \hat{k}\left(\frac{D\hat{p}}{d\tau}\right) = 0$$

$$\frac{d}{d\lambda}(\hat{k}\cdot\hat{p})$$

$$\propto \nabla_{\hat{p}}(\hat{k}\cdot\hat{p})$$

$$= p^\mu \nabla_\mu k_\nu p^\nu$$

$$= \underbrace{p^\mu(\nabla_\mu p^\nu)}_{0 \text{ (Geodesic)}} k_\nu + p^\mu p^\nu \nabla_\mu k_\nu$$

$$= p^\mu p^\nu \underbrace{\nabla_{(\mu} k_{\nu)}}_{\text{killing eq.}}$$

$$= 0$$

$$R^{1,3} \begin{cases} \hat{k} = \partial_t \longrightarrow \text{energy} \\ \hat{k} = \partial_x \longrightarrow \text{momentum} \\ \hat{k} = \partial_\phi \longrightarrow \text{angular - moment} \end{cases}$$

$k_\mu T^{\mu\nu} = J^\nu$

conservation flow?

$$\nabla_\nu J^\nu = \nabla_\nu (k_\mu T^{\mu\nu})$$

$$= T^{\mu\nu} \nabla_\nu k_\mu + k_\mu \nabla_\nu T^{\mu\nu}$$

$$= T^{\mu\nu} \nabla_{(\nu} k_{\mu)}$$

$$= 0$$

$$\int_V \nabla_\nu J^\nu = \int_S \sqrt{\gamma}\, n_\nu J^\nu .$$

Killing Tensor

$$\nabla_{(\mu} k_{\nu_1 \cdots \nu_n)} = 0$$

$$\Rightarrow p^{\nu_1 \cdots \nu_n} k_{\nu_1 \cdots \nu_n} \longrightarrow \text{conserved.}$$

Kerr BH?

Number of independent killing vector in n-dim space?

i) $\hat{X}, \hat{Y}$ are killing vectors.

$\longrightarrow [\hat{X}, \hat{Y}]$. killing vector.

Lie algebra

$\{\hat{X}\}$. isometry group.

ii) $\mathbb{R}^n$. n translation

Cartesian coordinate

$x^i . x^j \longrightarrow C_n^2 = \frac{n(n-1)}{2}$

in total: $\frac{n(n+1)}{2}$: maximally

symmetric space

properties: homogenus.

isotropic

same R at all locations.

constant curvature.

$$R_{\mu\nu\sigma\rho} = \frac{R}{n(n-1)} \left(g_{\mu\sigma} g_{\nu\rho} - g_{\nu\sigma} g_{\mu\rho} \right)$$

- Classification.

Riemann manifold i) $\mathbb{R}^n$. $R = 0$ flat

ii) S^n. $R > 0$.

radius a.

$R \propto \frac{1}{a^2}$ closed

iii) $\mathbb{H}^n$ $R < 0$.

Poincare half-space

$$ds^2 = \frac{l^2}{y^2}(dx^2 + dy^2).$$

$$R = -\frac{2}{l^2} < 0.$$

l: "radius" open

FRW universe

$$ds^2 = -dt^2 + a^2(t)\underbrace{[\qquad\qquad]}_{\text{symmetric in space}} \begin{cases} \mathbb{R} \\ S \\ \mathbb{H} \end{cases}$$

Pseudo-Riemannian

i) $\mathbb{R}^{1,n-1}$

ii) dS^n de Sitter spacetime $\Lambda > 0$

$a(t) = e^{H_0 t}$. in inflation (de Sitter phase)

$\downarrow$

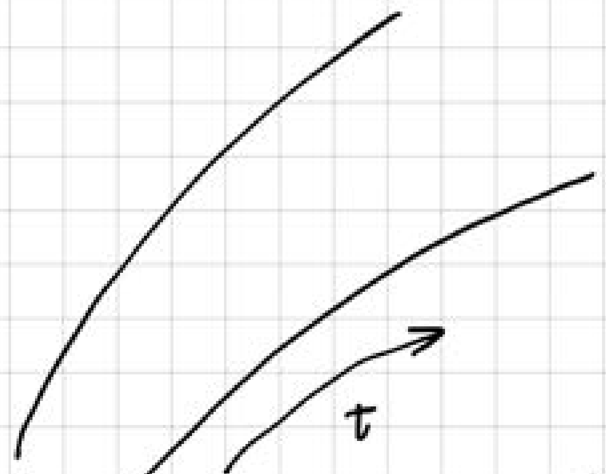
cosmological horizon

iii) AdS^n Anti de Sitter

AdS/CFT

Geodesic Deviation $\approx$ Tidal force

(s, t) surface filling



$\hat{T}$ ∂t. $\hat{S}$ ∂s

$[\partial t, \partial s] = 0$

$[\hat{T}, \hat{S}] = 0$.

change rate of $\hat{S}$ along geodesic?

$$\nabla_{\hat{T}} \nabla_{\hat{T}} \hat{S} = \frac{D^2}{dt^2} \hat{S}$$

i) Torsion free

$$\nabla_{\hat{T}} \hat{S} - \nabla_{\hat{S}} \hat{T} = 0.$$

$$\Rightarrow \nabla_{\hat{T}} \nabla_{\hat{S}} \hat{T} = 0$$

Notice $R(\hat{T}\ \hat{S}) = \nabla_{\hat{T}} \nabla_{\hat{S}} - \nabla_{\hat{S}} \nabla_{\hat{T}} - \nabla_{[\hat{T}, \hat{S}]}$

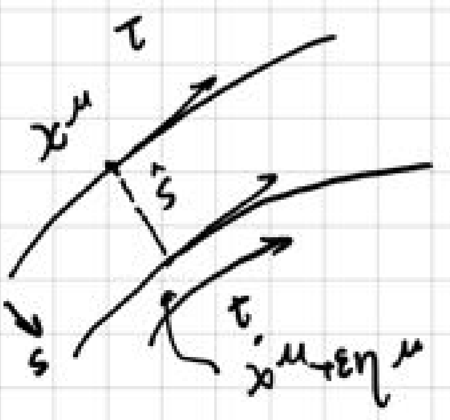
apply on $\hat{T}$. we have:

$$\nabla_{\hat{T}} \nabla_{\hat{S}} \hat{T} = R(\hat{T}\ \hat{S})\ \hat{T}$$

$$\Leftrightarrow \underbrace{\frac{D^2 S^\mu}{dt^2}}_{\hat{a}^\mu ?} = R^\mu{}_{\nu\sigma\rho} T^\sigma S^\rho T^\nu.$$

In Newtonian mechanics?

$$\frac{d^2 x^i}{dt^2} = a^i(x^\mu) = -\delta^{ij} \partial_j \Phi(x)$$



$$\frac{d^2(x^i+\varepsilon\eta^i)}{dt^2} = -\delta^{ij}\partial_j\Phi(x^i+\varepsilon\eta^i)$$

$$= -\delta^{ij}\partial_j\Phi - \delta^{ij}(\partial_j\partial_k\Phi)\varepsilon\eta^k$$

$$\Rightarrow \frac{d^2\eta^i}{dt^2} = \delta^{ij}(\partial_j\partial_k\Phi)\eta^k$$

$$= (\partial^i\partial_k\Phi)\eta^k$$

$$\equiv -K^i{}_j\eta^j$$

$$\Rightarrow \frac{d^2\eta^i}{dt^2} + K^i{}_j\eta^j = 0$$

$K^i{}_i = 0$ $\left(\partial^i\partial_i\Phi = 0, \text{ traceless}\right)$.

compare with

$$\frac{D^2}{dt^2}S^\mu = R^\mu{}_{\nu\sigma\rho}T^\sigma S^\rho T^\nu$$

$$\equiv -K^\mu{}_\rho S^\rho$$

Some note:

if $\hat{T}\cdot\hat{s} \neq 0$ at first?

Def, $\hat{\eta} = \hat{s} + \hat{T}(\hat{T}\cdot\hat{s})$

$$\hat{\eta}\cdot\hat{T} = \hat{s}\cdot\hat{T} + \underbrace{(\hat{T}\cdot\hat{T})}_{-1}(\hat{T}\cdot\hat{s})$$

$$= 0$$

substitude $\hat{s}$ for $\hat{\eta}$

$\hat{s} \perp \hat{\eta} \longrightarrow$ space component.

Tidal acceleration of a spherical symmetric mass.

$$a_{ij} = -\partial_i\partial_j\Phi = -(\delta_{ij} - 3n_in_j)\frac{GM}{r^3} \quad (n_i = \frac{x_j}{r})$$

$\downarrow$

(r, θ, ϕ)

$$a_{rr} = \frac{2GM}{r^3} \qquad a_{\phi\phi} = -\frac{GM}{r^3}$$

Observer LIF.

$\{\hat{e}_m\}$. $\hat{e}_0 = \hat{T}$ $\quad \hat{e}_0 \cdot \hat{e}_j = 0$ $\quad \hat{e}_i \cdot \hat{e}_j = 0$

$$\eta^m = e^m{}_\mu \eta^\mu$$

$\eta^0 = T_\mu \eta^\mu = 0$ $\quad \eta^i$. spatial.

$$\frac{D^2}{dt^2}\eta^i - \underbrace{R^\mu{}_{\nu\rho\sigma} e^i{}_\mu T^\nu \eta^\sigma T^\rho e^\sigma{}_j}_{\cdot K^i{}_j \eta^j} = 0$$

$$T^\sigma = (1 \quad 0 \quad 0 \quad 0)$$

$$e^\mu{}_i = \delta^\mu{}_i$$

$$\Rightarrow K^i{}_j = - R^\mu{}_{\nu\rho\sigma} e^i{}_\mu T^\nu \eta^\sigma T^\rho e^\sigma{}_j$$

$$= - R^\mu{}_{00\sigma} e^i{}_\mu T^0 T^0 \eta^\sigma e^\sigma{}_j$$

$$= - R^i{}_{00j}$$

In vaccum $K^i{}_i = R^j{}_{00j} = 0$

$$\longrightarrow R^\mu{}_{00\mu} = 0 \longrightarrow R^\mu{}_{\sigma\rho\mu} T^\sigma T^\rho = 0.$$

$\xrightarrow{\text{imply}} R_{\sigma\rho} T^{\sigma} T^{\rho} = 0.$

arbitany $T^{\sigma}. T^{\rho}.$ $R_{\sigma\rho} = 0.$

$\downarrow$

Vaccum field eq.

Newtonian approximation

i) weak field limit

$$g_{\mu\nu} = \eta_{\mu\nu} + \varepsilon h_{\mu\nu}, \quad \partial_0 h_{\mu\nu} = 0 \text{ (static)}$$

ii) non-relativistic particle $v \ll c$

$\left(\frac{dx^i}{d\tau} \ll \frac{dt}{d\tau} \right)$

$$g^{\mu\nu} = \eta^{\mu\nu} - \varepsilon h^{\mu\nu}. \quad h^{\mu\nu} = \eta^{\mu\rho} \eta^{\nu\sigma} h_{\rho\sigma}$$

$\Gamma^{\mu}_{\sigma\rho}$. keep $O(\epsilon)$

$$\Gamma^{\mu}_{\sigma\rho} = \frac{\epsilon}{2} \eta^{\mu\nu} (\partial_\sigma h_{\rho\nu} + \partial_\rho h_{\sigma\nu} - \partial_\nu h_{\sigma\rho})$$

$$\frac{d^2 x^{\mu}}{d\tau^2} + \underline{\Gamma^{\mu}_{\sigma\rho} \frac{dx^{\rho}}{d\tau} \frac{dx^{\sigma}}{d\tau}} = 0 \quad \rightarrow \Gamma^{\mu}_{00}$$

$$t \sim \tau. \quad \Gamma^{\mu}_{00} \sim \begin{cases} O(\epsilon^2) & \mu = 0 \\ \frac{1}{2} \epsilon \partial_i h_{00} & \end{cases}$$

$$\Rightarrow \begin{cases} \dfrac{d^2 t}{d\tau^2} = 0 \\ \dfrac{d^2 x^i}{d\tau^2} + \dfrac{1}{2}\epsilon\,\partial_i h_{00} \underbrace{\dfrac{dt}{d\tau}\dfrac{dt}{d\tau}}_{=1} = 0 \end{cases}.$$

compare with newtonian potential

$$\Rightarrow \Phi = \frac{1}{2}\epsilon h_{00}$$

or $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ $\quad$ ($h_{\mu\nu} \ll 1$)

$$\Rightarrow \Phi = -\frac{1}{2} h_{00}$$

So far we have take Einstein eq into account.

$R_{\mu\nu} = 0 \sim \nabla^2 \Phi = 0$?

consider $R_{00} = 0$

$$R_{00} = R^{\mu}{}_{0\mu 0}$$

$$R^{\mu}{}_{0\nu 0} = -\frac{1}{2}\eta^{\mu\sigma}\partial_\sigma \partial_\nu h_{00} + O(\epsilon)$$

$$\Rightarrow R_{00} = -\frac{1}{2}\Box h_{00} = -\frac{1}{2}\nabla^2 h_{00}$$

$\Rightarrow$ Laplace eq $\nabla^2 \Phi = 0$

To Einstein eq ?

$\nabla^2 \Phi = 4\pi G \rho \longrightarrow$ relativistic form

$$\rho \to \begin{cases} T_{\mu\nu} \, ? \\ T^{\mu}{}_{\mu} \, ? \end{cases}$$

a simple idea $\Rightarrow$ $\Box \Phi = T^{\mu}{}_{\mu}$

i) mismatch of observation

ii) 4D EM, $T^{\mu}{}_{\mu} = 0$ trivial

(no gravitational redshift).

$\rho \to T_{\mu\nu}$

$\Phi \to g_{\mu\nu}$ $\qquad \nabla^2 \to \nabla^{\mu}\nabla_{\mu}$, $\nabla_{\mu} g_{\sigma\rho} = 0$

$\nabla^2 \Phi$: 2^{nd} derivative

$\longrightarrow R_{\mu\nu} \sim T_{\mu\nu}$?

$\nabla_{\mu} T^{\mu\nu} = 0$, $\nabla_{\mu} R^{\mu\nu} \neq 0$

$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \qquad \nabla^{\mu} G_{\mu\nu} = 0$.

Hence we guess:

$$G_{\mu\nu} = \kappa T_{\mu\nu}$$

identify κ

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \kappa T_{\mu\nu}$$

$$R - \frac{1}{2} D R = \kappa T$$

$$\Rightarrow R = \frac{2\kappa T}{2 - D}$$

$$\therefore R_{\mu\nu} = \kappa \left(T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T \right)$$

weak field $R_{00} = -\frac{1}{2} \Box h_{00}$

let $T_{\mu\nu} = \begin{pmatrix} \rho & & & \\ & 0 & & \\ & & 0 & \\ & & & 0 \end{pmatrix}$

$$R_{00} = \kappa \left(\rho - \frac{1}{2} (-1) \cdot (-\rho) \right) = \frac{\kappa}{2} \rho$$

$$\parallel$$

$$\nabla^2 \Phi$$

$$\Rightarrow \kappa = 8\pi G$$

$$G_{\mu\nu} = 8\pi G T_{\mu\nu} \xleftrightarrow{\text{coupled}} \frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{\sigma\nu}\frac{dx^\sigma}{d\tau}\frac{dx^\nu}{d\tau} = 0$$

geometry matter.

due to compatible metric

$\nabla^\mu g_{\mu\nu} = 0$, to ensure the conservation of $T_{\mu\nu}$

$$(G_{\mu\nu} + \Lambda g_{\mu\nu}) = 8\pi G T_{\mu\nu}$$

$$[\Lambda] = L^{-2}$$

Λ cosmological constant

back to newtonian

$$\nabla^2 \Phi = 4\pi G \rho - \Lambda$$

$$\vec{g} = -\nabla\Phi = \left(-\frac{GM}{r^2} + \frac{\Lambda r}{3}\right)\vec{e}_r$$

repulsive

Weak field approximation (Linearized gravity)

$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, in RNC.

$$R_{\mu\nu\sigma\rho} = \frac{1}{2}\left(\partial_\rho\partial_\nu h_{\mu\sigma} + \partial_\sigma\partial_\mu h_{\nu\rho} - \partial_\rho\partial_\mu h_{\nu\sigma} - \partial_\nu\partial_\sigma h_{\mu\rho}\right) + O(\epsilon^2)$$

$$\Rightarrow R_{\nu\sigma} = \frac{1}{2}\Big(\partial^\mu \partial_\nu h_{\mu\sigma} + \partial_\sigma \partial^\rho h_{\nu\rho}$$

$$- \Box h_{\nu\sigma} - \partial_\nu \partial_\sigma h \Big) + O(\epsilon^2)$$

$$R = \eta^{\nu\sigma} R_{\nu\sigma}$$

$$= \frac{1}{2}\Big(\partial^\mu \partial^\sigma h_{\mu\sigma} + \partial^\nu \partial^\rho h_{\nu\rho} - \Box h - \Box h \Big)$$

$$= \partial_\mu \partial_\sigma h^{\mu\sigma} - \Box h$$

$G_{\mu\nu} \propto T_{\mu\nu} \sim O(\epsilon)$ $\quad \nabla^\mu T_{\mu\nu} = 0$

$$\Big\downarrow \text{(flat background)}$$

$$\partial^\mu T_{\mu\nu} = 0$$

$$x^\mu \mapsto x'^\mu$$

$$g'_{\mu'\nu'} = \frac{\partial x^\mu}{\partial x'^{\mu'}} \frac{\partial x^\nu}{\partial x'^{\nu'}} g_{\mu\nu}. \quad \text{s.t}$$

$$g'_{\mu\nu} = \eta_{\mu\nu} + h'_{\mu\nu}$$

let $x'^\mu = x^\mu + \xi^\mu$ $\qquad \xi^\mu \sim O(\epsilon)$

$$\frac{\partial x'^\mu}{\partial x^\nu} = \delta^\mu{}_\nu + \partial_\nu \xi^\mu$$

$$\downarrow \quad \frac{\partial x^\nu}{\partial x'^\mu} = \delta^\nu{}_\mu - \partial^\nu \xi_\mu$$

$$\eta_{\mu\nu} + h'_{\mu\nu} = \left(\delta^\sigma_\mu - \partial^\sigma \xi_\mu\right)\left(\delta^\rho_\nu - \partial^\rho \xi_\nu\right)$$

$$(\eta_{\sigma\rho} + h_{\sigma\rho})$$

$$= \eta_{\mu\nu} - \delta^\sigma_\mu (\partial^\rho \xi_\nu)\eta_{\sigma\rho}$$

$$- \delta^\rho_\nu (\partial^\sigma \xi_\mu)\eta_{\sigma\rho} + \delta^\sigma_\mu \delta^\rho_\nu h_{\sigma\rho}$$

$$= \eta_{\mu\nu} + h_{\mu\nu} - (\partial^\rho \xi_\nu)\eta_{\mu\rho} - (\partial^\sigma \xi_\mu)\eta_{\sigma\nu}$$

$$= h_{\mu\nu} - \partial_\mu \xi_\nu - \partial_\nu \xi_\mu$$

$R_{\mu\nu\sigma\rho} = R'_{\mu\nu\sigma\rho}$.

lifted by $\eta^{\mu\nu}$

Gauge choice $g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu}$

- Lorenz gauge $\partial^\mu A_\mu = 0 \rightarrow \Box A_\mu \sim J_\mu$

- Harmonic gauge

$\Box x^\mu = 0$

$$\Rightarrow \frac{1}{\sqrt{-g}} \partial_\nu \left(\sqrt{-g}\, g^{\nu\sigma} \partial_\sigma x^\mu\right) = 0$$

$$\Rightarrow \partial_\nu \left(\sqrt{-g}\, g^{\nu\mu}\right) = 0.$$

$$|-g| = \begin{vmatrix} -1+h_{00} & & & \\ & 1+h_{11} & & \\ & & 1+h_{22} & \\ & & & 1+h_{33} \end{vmatrix}$$

$$= -1 + h_{00} - h_{11} - h_{22} - h_{33} + O(\epsilon^2)$$

$$= -1 - h$$

$$(\partial_\nu \sqrt{-g})\, g^{\nu\mu} + \sqrt{-g}\, \partial_\nu g^{\nu\mu} = 0$$

$$= \frac{1}{2\sqrt{-g}}\, \partial_\nu(g)\, g^{\nu\mu} + \sqrt{-g}\, \partial_\nu h^{\nu\mu}$$

$$= \frac{1}{2}(\partial_\nu h)\, \eta^{\nu\sigma} - \partial_\nu h^{\nu\sigma} = 0.$$

trace

$$\Rightarrow \partial_\nu h^\nu{}_\rho - \frac{1}{2}\partial_\rho h = 0$$

Def "trace-reversed" fluctuation

$$\bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu} h$$

$$(\bar{h} = \eta^{\mu\nu}\bar{h}_{\mu\nu} = h - 2h = -h)$$

$$\partial_\nu \bar{h}^\nu{}_\rho = 0$$

$$\Box = \eta^{\mu\nu}\partial_\mu\partial_\nu$$

$$G_{\mu\nu} = -\frac{1}{2}\Box \bar{h}_{\mu\nu} \Rightarrow \Box \bar{h}_{\mu\nu} = -16\pi G T_{\mu\nu}$$

Other components?

ex:

$$T_{\mu\nu} = \begin{pmatrix} \rho & & & \\ & 0 & & \\ & & 0 & \\ & & & 0 \end{pmatrix}$$

$$\Box \bar{h}_{\mu\nu} = -16\pi G T_{\mu\nu} \rightarrow \bar{h}_{\mu\nu} \sim \begin{cases} \bar{h}_{00} \\ O(\epsilon^2) \end{cases}$$

$$\downarrow$$

$$\left.\begin{aligned} \nabla^2 \bar{h}_{00} &= -16\pi G\rho \\ \nabla^2 h_{00} &= -2\nabla^2\Phi = -8\pi G\rho \end{aligned}\right\} \Rightarrow \bar{h}_{00} = 2h_{00}$$

$$\bar{h}_{ij} = h_{ij} - \frac{1}{2}\delta_{ij} h$$

$$\Rightarrow h_{ij} = \frac{1}{2}\delta_{ij} h = \delta_{ij} h_{00} = -2\Phi\delta_{ij}$$

$$\therefore ds^2 = g_{\mu\nu} dx^\mu dx^\nu$$

$$= (-1 + h_{00}) dt^2 + (\delta_{ij} + h_{ij}) dx^i dx^j$$

$$= -(1+2\Phi) dt^2 + (1-2\Phi)\delta_{ij} dx^i dx^j$$

WEP $m_i = m_g$

$\sim \vec{a} = \vec{g}$

EEP LIF

how to couple to gravity?

ϕ. A_μ. ψ.

$\downarrow$

Minimal coupling principle.

$$\begin{cases} \eta_{\mu\nu} \mapsto g_{\mu\nu} \\ \partial_\mu \mapsto \nabla_\mu \end{cases}$$

terms like ϕR : non-minimal coupling

ex: i) $0 = \frac{d}{d\lambda}\left(\frac{dx^\mu}{d\lambda}\right) = \frac{dx^\nu}{d\lambda}\partial_\nu\left(\frac{dx^\mu}{d\lambda}\right)$

$$\mapsto \frac{dx^\nu}{d\lambda}\nabla_\nu\left(\frac{dx^\mu}{d\lambda}\right)$$

$$= \frac{dx^\nu}{d\lambda}\left(\partial_\nu \frac{dx^\mu}{d\lambda} + \Gamma^\mu_{\nu\sigma}\frac{dx^\sigma}{d\lambda}\right)$$

ii) $\partial_\mu T^{\mu\nu} = 0 \quad \longmapsto \quad \nabla_\mu T^{\mu\nu} = 0$

iii) EM: $\begin{cases} \partial_\mu F^{\mu\nu} = 4\pi J^\nu \\ \partial_{[\mu} F_{\nu\sigma]} = 0 \end{cases} \longmapsto \begin{matrix} \nabla_\mu F^{\mu\nu} = 4\pi J^\nu \\ \nabla_{[\mu} F_{\nu\sigma]} = 0 \end{matrix}$

backward to.

$\downarrow$ since

$$\begin{cases} d(*F) = 0 \\ dF = 0 \end{cases}$$

however, in some non-minimal coupling.

$$\partial_\mu \Big((1 + \alpha R) F^{\mu\nu} \Big) = 4\pi J^\nu$$

$$[\alpha] = L^{-2}$$

$$\alpha \propto l_p^2$$

$$l_p^2 \sim \frac{G\hbar}{c^3}$$

iv) some ambiguity

$$X_\mu \partial_\sigma \partial_\rho Y_\nu \longmapsto \begin{cases} X_\mu \nabla_\sigma \nabla_\rho Y_\nu \\ X_\mu \nabla_\rho \nabla_\sigma Y_\nu \end{cases} \sim R_{\mu\sigma\rho\nu}$$

v) Fermi field ?

$\psi\ \bar{\psi}$. spin interpretation of $SO(1,3)$

$\sim$ spin connection

$\{\hat{e_m}\}$ $\quad \hat{e_m}\cdot\hat{e_n} = \eta_{mn}$

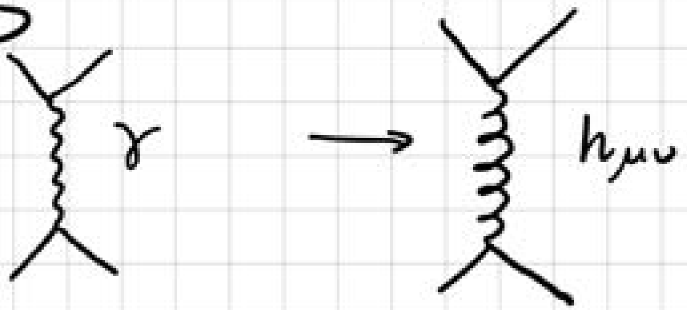
$G_{\mu\nu} = 8\pi G T_{\mu\nu}$

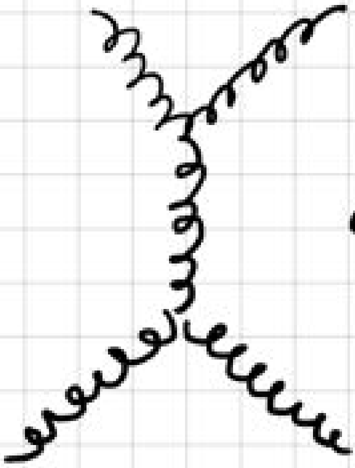
- 10 eqs. $-$ ($\nabla^\mu G_{\mu\nu} = 0$, 4 constraint)
  $\Rightarrow$ 6 dynamics eqs.
- $g_{\mu\nu}$. 10

  $-$ (gauge symmetry, 4) $\Rightarrow$ 6
  component related to dynamics

Non-linear theory.

QED





coupling of graviton.

Action principle $g_{\mu\nu} \mapsto S_{EH}$

. path integral $e^{iS/\hbar} \Rightarrow [S] = [\hbar]$

$$S = \int \sqrt{-g}\, d^4x\, \mathcal{L}$$

$\mathcal{L}$ should include: 2^{nd} derivative of the field.

like scalar field $\partial_\mu \phi \partial^\mu \phi$

EM. $F^{\mu\nu} F_{\mu\nu}$

$g_{\mu\nu} \longrightarrow R_{\mu\nu\sigma\rho}$ (2^{nd} derivative, but not scalar)

$$\Downarrow$$

R (R_2, R_3?) negligible at low energy.

$$\mathcal{L}_{EH} = R$$

if c.c included

$$\mathcal{L}_{EH} = R - 2\Lambda$$

$$S = K \int \sqrt{-g}\, d^4x\, \mathcal{L}_{EH}.$$

$$[K] \cdot L^2 = [\hbar]$$

$$[K] = [\hbar] \cdot L^{-2}$$

$$\sim \hbar \cdot l_p^{-2} = \frac{c^3}{G}$$

$$\therefore K \propto \frac{1}{G}$$

$$S = S_{EH} + S_{matter} \rightarrow T_{\mu\nu}$$

$$\frac{\delta S}{\delta g_{\mu\nu}} = 0 \longrightarrow \text{Einstein eq.}$$

$$S_{EH} = \frac{1}{16\pi G}\int \sqrt{-g}\, d^4x\, R$$

- $\phi \rightarrow \delta\phi$
- $g_{\mu\nu} \longrightarrow R^{\mu\nu}$. instead, we use $\delta g^{\mu\nu}$

① $\delta\sqrt{-g} = -\frac{\sqrt{-g}}{2}\, g_{\mu\nu}\,\delta g^{\mu\nu}$

② $\delta R = \delta(g^{\mu\nu}R_{\mu\nu})$

$$= R_{\mu\nu}\,\delta g^{\mu\nu} + g^{\mu\nu}\,\delta R_{\mu\nu}$$

$$R_{\mu\nu} = R^{\sigma}{}_{\mu\sigma\nu}$$

Polatini formulism $\frac{\delta R_{\mu\nu}}{\delta\Gamma}\frac{\delta\Gamma}{\delta g_{\mu\nu}}$

$$\delta R^{\alpha}{}_{\beta\gamma\lambda} = \nabla_{\gamma}(\delta\Gamma^{\alpha}_{\lambda\beta}) - \nabla_{\lambda}(\delta\Gamma^{\alpha}_{\beta\gamma})$$

boundary terms. $\delta R_{\mu\nu} = 0$.

$$\therefore \delta S_{EH} = \frac{1}{16\pi G}\int d^4x \left(-\frac{\sqrt{-g}}{2}\, g_{\mu\nu}R + \sqrt{-g}\, R\right)\delta g^{\mu\nu}$$

$$S_{matter} = \int d^4x \, \mathcal{L}_{matter}$$

$$T_{\mu\nu} \equiv -\frac{2}{\sqrt{-g}} \frac{\delta \mathcal{L}_{matter}}{\delta g^{\mu\nu}}$$ (symmetric)

$$\therefore \delta S_{matter} = -\frac{1}{2} \int d^4x \sqrt{-g} \, T_{\mu\nu} \, \delta g^{\mu\nu}$$

$$\therefore \delta S = \int \sqrt{-g} \, d^4x \cdot \frac{1}{2} \cdot \left(\frac{1}{8\pi G} G_{\mu\nu} - T_{\mu\nu} \right) \delta g^{\mu\nu} = 0$$

$$\Rightarrow G_{\mu\nu} = 8\pi G \, T_{\mu\nu}$$

Examples:

1. Scalar field:

$$S_{scalar} = \int d^4x \sqrt{-g} \left(-\frac{1}{2} g^{\mu\nu} \nabla_\mu \phi \nabla_\nu \phi - V(\phi) \right)$$

$$\delta \mathcal{L}_{matter} = \int d^4x \left(-\frac{\sqrt{-g}}{2} g_{\mu\nu} \left(-\frac{1}{2} g^{\mu\nu} \nabla_\mu \phi \nabla_\nu \phi \right.\right.$$
$$\left.\left. - V(\phi) \right) + \sqrt{-g} \left(-\frac{1}{2} \nabla_\mu \phi \nabla_\nu \phi \right) \right) \delta g^{\mu\nu}$$

$$\Rightarrow T_{\mu\nu} = g_{\mu\nu} \left(-\frac{1}{2} (\nabla \phi)^2 - V(\phi) \right) + \nabla_\mu \phi \nabla_\nu \phi$$

flat spacetime $\mathcal{H} = \frac{1}{2} (\dot{\phi})^2 + \frac{1}{2} (\vec{\nabla} \phi)^2 + V(\phi)$

2. EM field A_μ

$$S_{EM} = \int d^4x \sqrt{-g} \left(-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} \right) \qquad F^{\mu\nu} \to g^{\mu\sigma} g^{\nu\rho} F_{\sigma\rho}.$$

$$\delta S_{EM} = \int d^4x \left(-\frac{\sqrt{-g}}{2} g_{\mu\nu} \left(-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} \right) \delta g_{\mu\nu} \right)$$

$$-\frac{1}{4}\sqrt{-g}\,\delta \left(\underline{F_{\sigma\rho}\, g^{\sigma\kappa} g^{\rho\lambda} F_{\kappa\lambda}} \right)$$

$$= F_{\sigma\rho} F_{\kappa\lambda} \left(\delta^\sigma_\mu \delta^\kappa_\nu g^{\rho\lambda} + \delta^\rho_\mu \delta^\lambda_\nu g^{\sigma\kappa} \right)$$

$$= F_{\mu\rho} F_{\nu\lambda} g^{\rho\lambda} + F_{\mu\sigma} F_{\nu\kappa} g^{\sigma\kappa}$$

$$= 2 F_{\mu\rho} F_{\nu\lambda} g^{\rho\lambda}$$

$$= 2 F_\mu{}^\lambda F_{\nu\lambda}$$

$$\Rightarrow T_{\mu\nu} = -F_\mu{}^\lambda F_{\lambda\nu} - \frac{1}{4} g_{\mu\nu} F^2$$

$$T^\mu{}_\mu = -F_\mu{}^\lambda F_\lambda{}^\mu - \frac{D}{4} F^2$$

$$= \left(1 - \frac{D}{4} \right) F^2 \qquad D=4 \text{ traceless. scale inv.}$$

3. c.c

$$S_{cc} = \frac{1}{16\pi G}\int d^4x\sqrt{-g}\,(-2\Lambda)$$

$$\delta S_{cc} = \frac{1}{16\pi G}\int d^4x \left(\frac{\sqrt{-g}}{2}\, g_{\mu\nu}\,(2\Lambda)\,\delta g_{\mu\nu}\right)$$

$$\therefore\ \delta S_{EH} + \delta S_{cc} = \frac{1}{16\pi G}\int d^4x\sqrt{-g}\,\left(G_{\mu\nu} + \Lambda g_{\mu\nu}\right)\delta g_{\mu\nu}$$

$$T_{\mu\nu} = -\frac{2}{\sqrt{-g}}\left(-\frac{\Lambda}{8\pi G}\right)\left(-\frac{1}{2}\sqrt{-g}\right) g_{\mu\nu}$$

$$= -\frac{\Lambda}{8\pi G}\, g_{\mu\nu}$$

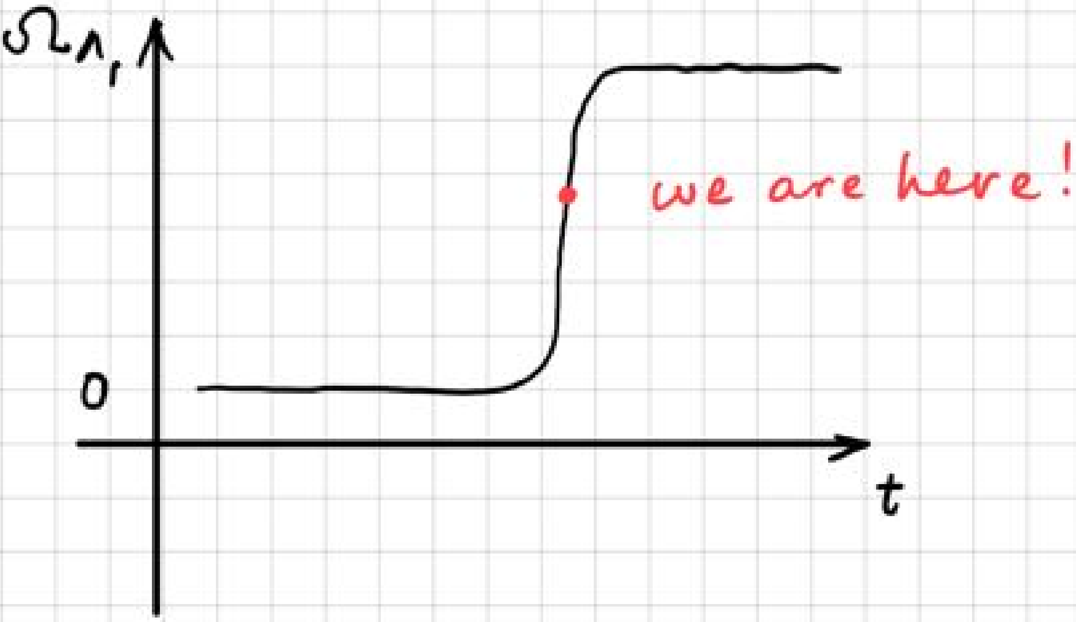
$$= -\rho_c\, g_{\mu\nu}$$

compare with $T_{\mu\nu}$

$$T_{\mu\nu} = (\rho + p)\,u_\mu u_\nu + p g_{\mu\nu}$$

$\Rightarrow\ \rho_\Lambda + p_\Lambda = 0 \quad \therefore p = -\rho \quad w = -1$

ZPE? too large $E = \frac{1}{2}\hbar\omega$ ~ tens of orders of magnitude coincidence? we are at a "special point"



Point particle:

$$S = -m\int d\tau = -m\int\sqrt{-g_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}}\, d\lambda$$

$\Rightarrow$ geodesic eq.

$T_{\mu\nu} = ?$

- line integral $\longrightarrow$ volume integral?

$$1 = \int d^4y\, \delta^{(4)}\left(y^\mu - x^\mu(\lambda)\right)$$

$$\therefore S = -m\int d^4y\, d\lambda \underbrace{\sqrt{-g_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}}\,\delta^{(4)}\left(y^\mu - x^\mu(\lambda)\right)}_{\text{treated as } \mathcal{L}}$$

$$\therefore T_{\mu\nu} = -\frac{2m}{\sqrt{-g}} \int d\lambda \frac{1}{2\sqrt{-\dot{x}^2}} (-\dot{x}_\mu \dot{x}_\nu) \delta^{(4)}(y^\mu - x^\mu(\lambda))$$

$$\nabla_\mu T^{\mu\nu} = \frac{1}{\sqrt{-g}} \partial_\mu (\sqrt{-g}\, T^{\mu\nu}) + \Gamma^\nu_{\mu\sigma} T^{\mu\sigma}$$

$$= \int d\lambda\, \dot{x}^\mu \dot{x}^\nu \frac{\partial}{\partial y^\mu} \delta^{(4)}(y^\mu - x^\mu(\lambda))$$

$$+ \Gamma^\nu_{\sigma\rho} \int d\lambda\, \dot{x}^\sigma \dot{x}^\rho \delta^{(4)}(y^\mu - x^\mu(\lambda)) = 0$$

$$\Rightarrow \int d\lambda \frac{d}{d\lambda} (\dot{x}^\nu) \delta^{(4)}(y^\mu - x^\mu)$$

$$+ \int d\lambda\, \Gamma^\nu_{\sigma\rho} \frac{dx^\sigma}{d\lambda} \frac{dx^\rho}{d\lambda} \delta^{(4)}(y^\mu - x^\mu)$$

Einstein eq + $\nabla_\mu T^{\mu\nu} = 0$

$\longrightarrow$ geodesic eq.

$\nabla_\mu T^{\mu\nu} = 0 \Leftarrow$ Diffeomorphism (EOM of matter).

$x^\mu \longrightarrow x^\mu + \xi^\mu$

$g_{\mu\nu} \longrightarrow g_{\mu\nu} + \partial_\mu \xi_\nu + \partial_\nu \xi_\mu$

· Requirements on $T^{\mu\nu}$.

· Given $T_{\mu\nu} \Rightarrow$ implication to spacetime

Energy condition

weak energy condition (WEC)

$$T_{\mu\nu} t^\mu t^\nu \geq 0 \quad \forall \hat{t} \text{ timelike}$$

For perfect fluid

$$T_{\mu\nu} = (\rho + p) u_\mu u_\nu + p g_{\mu\nu}$$

u comoving velocity.

$\Rightarrow \rho \geq 0, \quad (\rho + p) \geq 0$

pf: $\hat{t} = a\hat{u} + b\hat{l}$ $\quad \hat{l}$: null $\hat{l} \cdot \hat{l} = 0$.

$$T_{\mu\nu}(a u^\mu + b l^\mu)(a u^\nu + b l^\nu)$$

$$= a^2 T_{\mu\nu} u^\mu u^\nu + ab T_{\mu\nu}(u^\mu l^\nu + u^\nu l^\mu) + b^2 T_{\mu\nu} l^\mu l^\nu$$

$$\begin{cases} T_{\mu\nu} u^\mu u^\nu = (p+\rho) + p(-1) = \rho \\ T_{\mu\nu} u^\mu l^\nu = -(\rho+p)(\hat{l}\cdot\hat{u}) + p(\hat{l}\cdot\hat{u}) = -\rho(\hat{l}\cdot\hat{u}) \\ T_{\mu\nu} l^\mu l^\nu = (\rho+p)(\hat{l}\cdot\hat{u})^2 \end{cases}$$

$$= a^2\rho - 2ab\rho(\hat{l}\cdot\hat{u}) + b^2(\rho+p)\cdot(\hat{l}\cdot\hat{u})^2$$

notice $\hat{t}\cdot\hat{t} < 0$

$$(a\hat{u}+b\hat{l})(a\hat{u}+b\hat{l})$$

$$= -a^2 + 2ab(\hat{l}\cdot\hat{u}) < 0$$

$$= \rho\left(a^2 - 2ab(\hat{l}\cdot\hat{u})\right) + b^2(p+\rho)(\hat{l}\cdot\hat{u})^2$$

$\therefore \rho \geq 0, \quad p+\rho \geq 0.$

ii) Null energy condition

$T_{\mu\nu} l^\mu l^\nu \geq 0 \quad \forall$ null vector $\hat{l}$

perfect fluid $\Rightarrow \rho + p \geq 0$

iii) Dominant energy condition

$$\text{WEC} + \underline{|-T_{\mu\nu} t^\nu| \leq 0}$$

energy flux future directed

non-spacelike

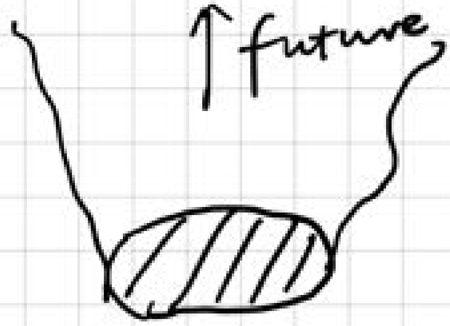
$$[(\rho+p)u_\mu u_\nu + p g_{\mu\nu}][$$

Causal Structure

causal curve $\rightarrow \begin{cases} \text{time like} \\ \text{null} \end{cases}$

$S \in M$

$J^+(S)$ (causal future)



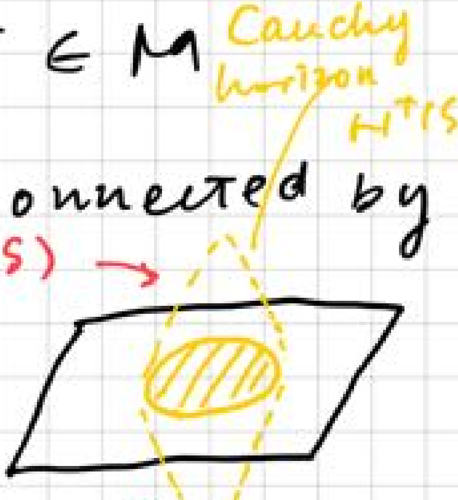
$I^+(S)$ chronological future

$I^+(S) \subset J^+(S)$. (time like curve connected)

a given

S: achronal closed set $\in M$

(no two points can be connected by time like curve)



$D^+(S)$ future domain of dependence

$D^-(S)$ past domain of dependence.

$D(S) = D^+(S) \cup D^-(S)$

$D(\Sigma) = M$. Σ. Cauchy surface.

↳ cover all manifold.

Σ can be easy to find in flat spacetime

however, in curved spacetime,

Not so easy because of the ill spacetime

- $\exists$ closed timelike curve
- Singularity.

Solutions to EFE

Stationary & Static.

$\downarrow$ $\frac{\partial}{\partial t} = 0$?

$\downarrow$ no motion.

we want our definition to be coordinate independent.

· Stationary spacetime

$\partial_0 g_{\mu\nu} = 0$ $\leftarrow$ this statement depends on coordinate.

$\Rightarrow \exists$ time like killing vector

$\downarrow$

coordinate independent.

$\downarrow$ in a specific coordinate

$\hat{\xi} = \partial t \qquad \xi^{\mu} = (1\ 0\ 0\ 0)$

example: kerr spacetime

· Static Spacetime

example: Schwarzchild spacetime.

$$ds^2 = g_{00}dt^2 + 2g_{0i}\,dt\,dx^i + g_{ij}\,dx^i dx^j$$

inv $t \to -t$

$\Rightarrow g_{0i} = 0$

$\downarrow$ translate into coordinate independent expression.

$\boxed{\exists \text{ time like } \hat{\xi} \perp \text{ hypersurface.}}$

$\Rightarrow \xi_{[\mu}\nabla_{\nu}\xi_{\rho]} = 0.$

Hypersurface. $\begin{cases} \text{spacelike if } \hat{n} \text{ timelike} \\ \text{timelike if } \hat{n} \text{ spacelike} \\ \text{null if } \hat{n} \text{ null} \end{cases}$

$f(x^\mu) = c$ (scalar function)

$\Rightarrow df$ 1-form.

$= \partial_\mu f\, dx^\mu$

$\partial_\mu f \longrightarrow n_\mu \equiv \dfrac{\partial_\mu f}{\|df\|}$

$\therefore \xi^\mu \propto n^\mu$

To solve highly non-linear equations

· Ansatz.

we suppose

S^2

$$ds^2 = -e^{2\alpha} dt^2 + e^{2\beta} dr^2 + r^2 d\Omega^2$$

$$d\Omega^2 = d\theta^2 + \sin^2\theta \, d\phi^2$$

spherically symmetric in 4-D?

killing vector field

S^2 $\quad \frac{d(d+1)}{2} = 3.$

$\hookrightarrow SO(3) \longrightarrow$ generator

$\downarrow$

$\hat{v}^{(1)} \quad \hat{v}^{(2)} \quad \hat{v}^{(3)}$

$\downarrow$

submanifold

maximally symmetric spacetime.

$[\hat{v}^1, \hat{v}^2] = 0$

Foliation

$\hookrightarrow v^2$

$$ds^2 = g_{ij} \, dv^i dv^j + f(v) \gamma_{ij} \, du^i du^j$$

example: R^3 : $ds^2 = d\rho^2 + \rho^2 d\Omega^2$
$$\downarrow$$
foliation.

$4D \to S^2$

(x^a, x^b, θ, ϕ)

$$ds^2 = g_{aa}\, dx^a dx^a + 2 g_{ab}\, dx^a dx^b + g_{bb}\, dx^b dx^b$$
$$+ r^2(x^a, x^b)\, d\Omega^2$$

$(x^a, x^b) \longmapsto (a, r)$

$$ds^2 = g_{aa}\, dx^a dx^a + 2 g_{ar}\, dx^a dx^r$$
$$+ g_{rr}\, dx^r dx^r + r^2 d\Omega^2$$

g_{aa}. g_{ar}. g_{rr} function of (x^a, r).

$$(x^a, r) \xmapsto{\text{Diagonalize}} (t, r)$$

Pf: $ds^2 = g_{tt}\, dt^2 + g_{rr}\, dr^2 + r^2 d\Omega^2$
$$\uparrow$$
$$dt = \frac{\partial t}{\partial x^a} dx^a + \frac{\partial t}{\partial r} dr$$

enough degrees of freedom to determine
$t = t(x^a, r)$.

$$ds^2 = m(t,r)\,dt^2 + n(t,r)\,dr^2 + r^2 d\Omega^2$$

some physical constraints.

$r \to \infty$, back to flat spacetime.

$$ds^2 \longrightarrow -dt^2 + dr^2 + r^2 d\Omega^2$$

$$m(t,r) = -e^{2\alpha(t,r)}$$

$$n(t,r) = e^{2\beta(t,r)}.$$

$\Big\downarrow$ calculate Ricci Tensor.

$$R_{tt} = [\partial_0^2 \beta + (\partial_0 \beta)^2 - \partial_0 \alpha \partial_0 \beta]$$

$$+ e^{2(\alpha-\beta)}\left[\partial_1^2 \alpha + (\partial_1 \alpha)^2 - \partial_1 \alpha \partial_1 \beta + \frac{2}{r}\partial_1 \alpha\right]$$

$$R_{rr} = -\left(\partial_1^2 \alpha + (\partial_1 \alpha)^2 - \partial_1 \alpha \partial_1 \beta + \frac{2}{r}\partial_1 \beta\right)$$

$$+ e^{2(\beta-\alpha)}[\partial_0^2 \beta + (\partial_0 \beta)^2 - \partial_0 \alpha \partial_0 \beta].$$

$$R_{tr} = \frac{2}{r}\partial_0 \beta$$

$$R_{\theta\theta} = e^{-2\beta}[r(\partial_1 \beta - \partial_1 \alpha) - 1] + 1$$

$$R_{\phi\phi} = \sin^2\theta \, R_{\theta\theta}$$

$R_{tr} = 0 \Rightarrow \beta = \beta(r)$

$\partial_0 R_{\theta\theta} = 0 \Rightarrow \partial_0 \partial_1 \alpha = 0 \Rightarrow \alpha = f(r) + g(t).$

$$g_{tt} = e^{2f(r)} \underbrace{e^{2g(t)} dt^2}_{\downarrow \; d\tilde{t}^2}$$

$\therefore ds^2 = -e^{2\alpha(r)} dt^2 + e^{2\beta(r)} dr^2 + r^2 d\Omega^2$

consider

$R_{tt} + e^{2(\alpha-\beta)} R_{rr} = 0$

$\Rightarrow \frac{2}{r} \partial_1 (\alpha + \beta) = 0.$

$\alpha + \beta = \text{const}.$

$\alpha = -\beta + c$ choose $c = 0$ (redefine t).

$\Rightarrow \alpha = -\beta.$

$R_{\theta\theta} = 0$

$\Rightarrow e^{2\alpha} \left(\partial_1 (-2\alpha) - 1 \right) + 1 = 0.$

$\partial_1 \left(r e^{2\alpha} \right) = 1$

$\Rightarrow e^{2\alpha} = 1 + \frac{c}{r}, \quad e^{2\beta} = \left(1 + \frac{c}{r} \right)^{-1}$

$$\therefore ds^2 = -\left(1+\frac{c}{r}\right)dt^2 + \left(1+\frac{c}{r}\right)^{-1}dr^2$$
$$+r^2 d\Omega^2$$

in Newtonian approximation.

$$h_{00} = -2\Phi, \quad \Phi = -\frac{GM}{r}.$$

here $h_{00} = g_{00}+1 = -\frac{c}{r}$

$\therefore c = -2GM.$ (if $c>0$, naked singularity).

$\Rightarrow$ Schwarzchild metric

$$ds^2 = -\left(1-\frac{2GM}{rc^2}\right)dt^2 + \left(1-\frac{2GM}{rc^2}\right)^{-1}dr^2$$
$$+r^2 d\Omega^2$$

properties:

1. Spherically symmetric
2. $r > 2GM$. stationary & static

   (only if $r > 2GM$. $\hat{\xi} = \partial_t$ time like).
3. coordinate singularity at Schwarzchild radius

   $r = R_s = \frac{2GM}{c^2}$ $\quad g_{tt} = 0.$

Black Hole. $\hat{\xi} = \partial_t$ is always a killing vector.

$$\|\hat{\xi}\| = g_{\mu\nu}\xi^\mu \xi^\nu$$
$$= -\left(1 - \frac{2GM}{r}\right)$$

$r > R_s$ $\|\hat{\xi}\| < 0$, time like.

$r = R_s$ $\|\hat{\xi}\| = 0$, null

$r < R_s$ $\|\hat{\xi}\| > 0$, space like.

$r \leq R_s$. $\hat{\xi}$ not time like.

$\Rightarrow$ not stationary.

4. Asymtopically flat.

$r \to \infty$ $ds^2|_{sch} \to ds^2|_{min}$

5. (t, r, θ, ϕ) Schwartzchild coordinate. $\longrightarrow$ static observer at infinity.

Birkhoff Theorem.

Schwarzchild spacetime is the unique spherically symmetric spacetime of vaccum Einstein equations.

implication:

Even if the source is evolving in spherically symmetric, the external observor will still see a Schwarzchild spacetime.

Israel Theorem

Schwarzchild spacetime is the unique static solution of vaccum EFE.

6) killing symmetric

$$SO(3) \times U(1)$$

$\downarrow$ $\qquad$ $\downarrow$

S^2 $\qquad$ ∂t

7) intrinsic singularity.

$R_{\mu\nu\sigma\rho}$. $\qquad$ $R_{\mu\nu} = 0 \begin{cases} \to R = 0 \\ \to R_2 = 0. \end{cases}$

$\downarrow$

$$R_4 = \frac{12 R_S^2}{r^6}$$

$r \to 0$ diverge

$\Rightarrow$ intrinsic singularity at $r = 0$.

protect by event horizon.

(if $c > 0$, naked singularity).

Geodesic in Schwarzchild particle.

massive

$$\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{\rho\sigma}\frac{dx^\rho}{d\tau}\frac{dx^\sigma}{d\tau} = 0$$

massless $\lambda \sim a\lambda + b$. affine parameterization.

Here we do not directly solve $\Gamma^\mu_{\rho\sigma}$ and geodesic eq.

Start from conserved quantities.

$\exists$ Killing vector field $\hat{k}$.

$\hat{k}\cdot\hat{p}$ is conserved

In Schwarzchild spacetime

$SO(3) \times U(1)$

$$\begin{cases} \hat{\xi} & \partial t & \text{time like} & (1\ 0\ 0\ 0) \\ \hat{\eta} & \partial\phi & \text{space like} & (0\ 0\ 0\ 1) \end{cases}$$

Massive particle $\hat{p} = m\hat{u}$

def. $E \equiv -\hat{\xi} \cdot \hat{u}$

$$= -g_{tt} u^t$$

$$= \left(1 - \frac{R_s}{r}\right) \frac{dt}{d\tau}$$

E: energy density observed by a observor at infinity.

$$\hat{u_0} = \hat{\xi} = (1 \quad 0 \quad 0 \quad 0)$$

$$\begin{cases} E = 1, & \text{static at } \infty \\ E \geq 1, & \text{can reach } \infty \end{cases}$$

similarly for $\hat{\eta}$

SO(3) → particle moving in a plane
↓
selected as equatorial plane. $\theta = \frac{\pi}{2}$?

$$L = \hat{\eta} \cdot \hat{u}$$

$$= g_{\phi\phi} \eta^{\phi} u^{\phi}$$

$$= r^2 \sin^2\theta \frac{d\phi}{d\lambda}$$

— Angular momentum density.

can we use $\theta = \frac{\pi}{2}$? (use geodesic eq of θ)

$$\frac{d^2\theta}{d\lambda^2} + \frac{2}{r}\frac{d\theta}{d\lambda}\frac{dr}{d\lambda} - \sin\theta\cos\theta\left(\frac{d\phi}{d\lambda}\right)^2 = 0.$$

$$\frac{1}{r^2}\frac{d}{d\lambda}\left(r^2\frac{d\theta}{d\lambda}\right) - \frac{\cos\theta}{r^2\sin\theta}L^2 = 0.$$

$$\left(r^2\frac{d\theta}{d\lambda}\right)^2 = -L^2\cot^2\theta + C.$$

initial condition $\begin{cases} \theta\big|_{t=0} = \frac{\pi}{2} \\ \frac{d\theta}{d\lambda}\Big|_{t=0} = 0. \end{cases}$

$$\Rightarrow \left(r^2\frac{d\theta}{d\lambda}\right)^2 + L^2\cot^2\theta = 0.$$

always satisfy $\longrightarrow \theta \equiv \frac{\pi}{2}$.

so we can let θ to be always $\frac{\pi}{2}$

$$\Rightarrow L = r^2\frac{d\phi}{d\lambda}$$

$$u^\mu = \left(\frac{dt}{d\lambda} \quad \frac{dr}{d\lambda} \quad 0 \quad \frac{d\phi}{d\lambda}\right)$$

$$u_\nu = (-E, \ \cdots \ 0, \ L)$$

note $\hat{u}\cdot\hat{u} = -k \begin{cases} = -1 & \text{massive} \\ = 0 & \text{massless} \end{cases}$

$$\Rightarrow -\left(1-\frac{R_s}{r}\right)\left(\frac{dt}{d\lambda}\right)^2 + \left(1-\frac{R_s}{r}\right)^{-1}\left(\frac{dr}{d\lambda}\right)^2$$

$$+ r^2\left(\frac{d\phi}{d\lambda}\right)^2 = 0.$$

$$\Rightarrow -E^2 + \left(\frac{dr}{d\lambda}\right)^2 + r^2\left(1-\frac{R_s}{r}\right)\left(\frac{d\phi}{d\lambda}\right)^2 = -k$$

$$-E^2 + \left(\frac{dr}{d\lambda}\right)^2 = -\left(1-\frac{R_s}{r}\right)\left(k+\frac{L^2}{r^2}\right)$$

$$\Rightarrow \frac{1}{2}\left(\frac{dr}{d\lambda}\right)^2 + \frac{1}{2}\left(1-\frac{R_s}{r}\right)\left(k+\frac{L^2}{r^2}\right) = \frac{1}{2}E^2$$

$$\text{Def: } V(r) \equiv \underbrace{\frac{k}{2}}_{\substack{\text{a const} \\ \text{trivial}}} - \underbrace{\frac{kGM}{r}}_{\substack{\text{newtonian} \\ \text{potential}}} + \underbrace{\frac{L^2}{2r^2}}_{\substack{\text{centri-} \\ \text{fugal} \\ \text{potential}}} - \underbrace{\frac{GML^2}{r^3}}_{\substack{\text{GR} \\ \text{correction}}}$$

$\downarrow$

preccession?

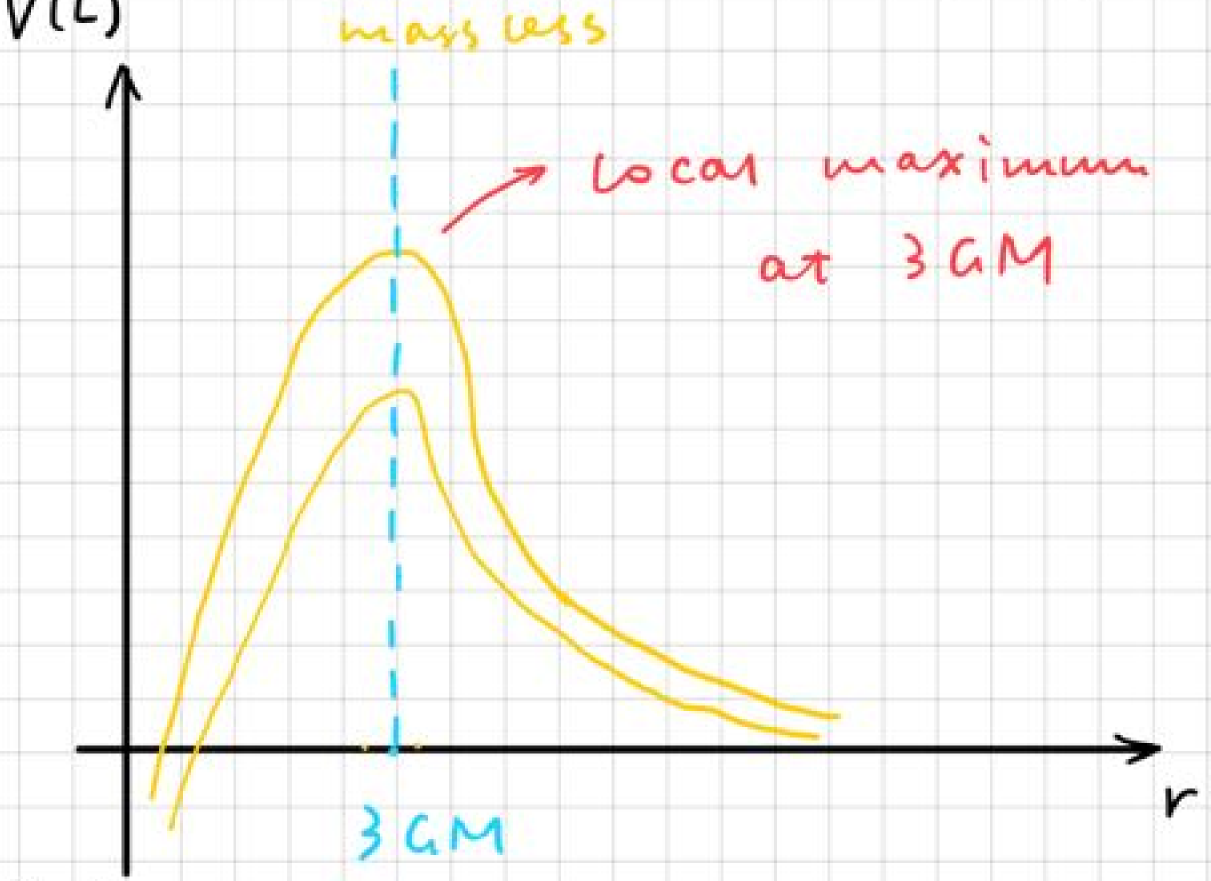
dominate

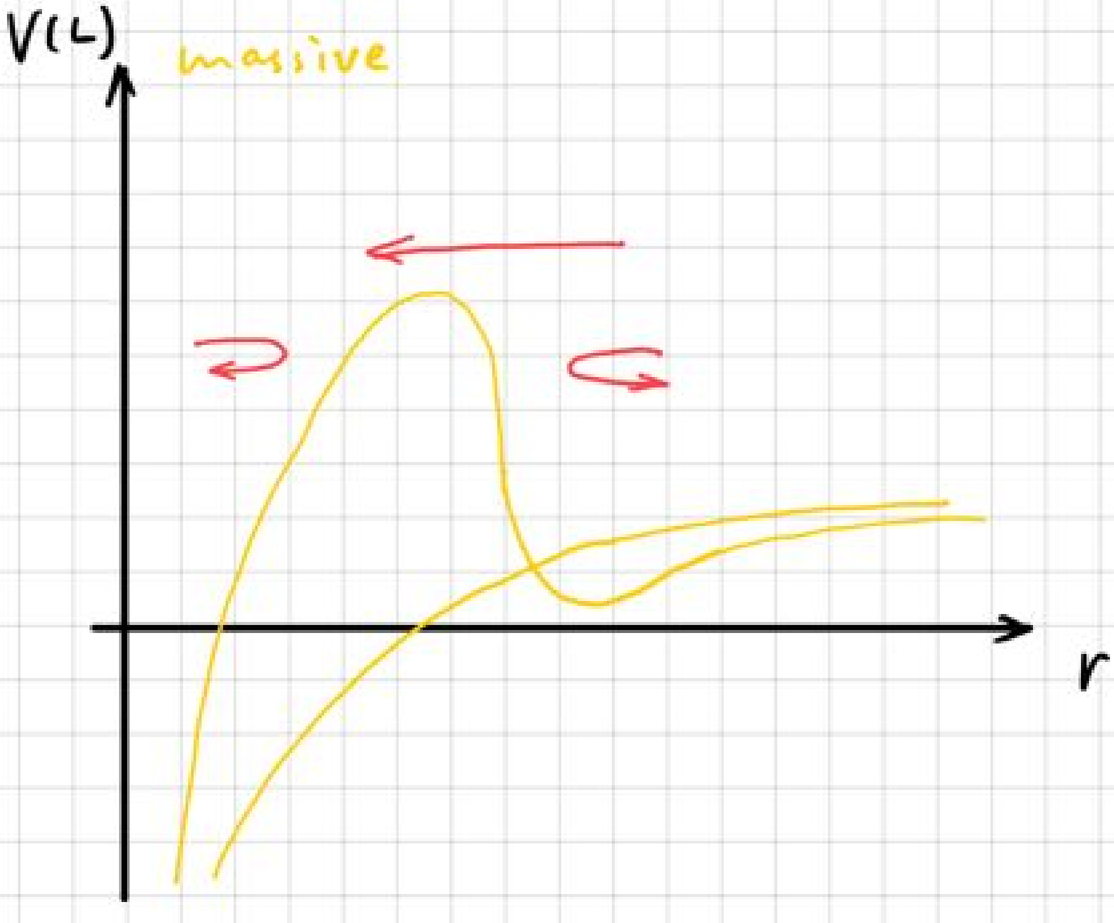
at $r \to 0$

Effectively → 1D problem.

$$\frac{dr}{d\lambda} = \pm\sqrt{\cdots} \quad \begin{cases} + & \text{outgoing} \\ - & \text{ingoing.} \end{cases}$$

bounded?
unbounded?
fall to the star?





Circular motion? $(k=1)$.

$$\frac{dV}{dr} = \frac{GM}{r^2} - \frac{L^2}{r^3} + \frac{3GML^2}{r^4}$$

$$\Rightarrow R_\pm = \frac{L^2 \pm \sqrt{L^4 - 12G^2M^2L^2}}{2GM}$$

only if $L^4 - 12G^2M^2L^2 \geq 0$.

$R_+ \geq 6GM$

$R_- \leq 6GM$, $L \gg GM$, $R_- \to 3GM$

$$\Rightarrow \begin{cases} R_+ \geq 6GM \\ 3GM \leq R_- \leq 6GM. \end{cases}$$

$$\frac{1}{2}E^2 = V(r = R_c)$$

$$= \frac{1}{2}\frac{(R_c - 2GM)^2}{R_c(R_c - 3GM)}$$

if $3GM \leq R_- \leq 4GM$, $E^2 \geq 1 \longrightarrow$ can be unbounded

$4GM \leq R_- \leq 6GM$, $E^2 < 1 \longrightarrow$ if outgoing. still bounded.

$L^2 = 12G^2M^2 \Rightarrow R_+ = R_- = 6GM$

Inner Most Stable Circular Orbit (ISCO)

$$\mathcal{E}^2 = \frac{(4GM)^2}{18(GM)^2} = \frac{8}{9}.$$

a particle freed at ∞. $\mathcal{E}^2 = 1$

to reach ISCO. energy loss $1 - \sqrt{\frac{8}{9}} \approx 5.7\%$

In circular orbit $\qquad L = R_c \left(\frac{GM}{R_c - 3GM} \right)^{1/2}$

$L \rightleftarrows R \to \mathcal{E} \to$ 4-velocity.

$$u^\mu = \left(\frac{dt}{d\lambda}, 0, 0, \frac{d\phi}{dt} \right)$$

$$\Rightarrow \frac{L}{\mathcal{E}} = \frac{r^2 \frac{d\phi}{d\lambda}}{\frac{dt}{d\lambda}\left(1 - \frac{R_s}{r}\right)} = \frac{r^2}{1 - \frac{R_s}{r}} \frac{d\phi}{dt}$$

$$= \frac{r^2}{1 - \frac{R_s}{r}} \Omega$$

angular velocity observed at ∞.

$$\Rightarrow \Omega^2 R_c^3 = \frac{R_s}{2}$$

$\to$ 3rd Law of Kepler.

$$u^\mu = \frac{dt}{d\tau}\left(1 \quad 0 \quad 0 \quad \Omega\right)$$

$$\Rightarrow \quad \frac{dt}{d\tau} = \left(1 - \frac{3GM}{Rc}\right)^{1/2}.$$

1) fixed particle: (Hanging observor)

$$u^\mu = \left(\frac{dt}{d\tau}, 0, 0, 0\right)$$

$$\Rightarrow \quad \left(\frac{dt}{d\tau}\right)^2 \left(1 - \frac{Rs}{r}\right) = 1$$

$$\therefore \quad \left(\frac{dt}{d\tau}\right) = \left(1 - \frac{Rs}{r}\right)^{1/2}$$

Hanging observor $\xrightarrow{\text{observor}}$ a particle at Rc.

$\frac{1}{\sqrt{1-v^2}}$ locally flat.

$$\gamma = -\hat{u}_0 \cdot \hat{u}^\mu = \left(1 - \frac{2GM}{Rc}\right)^{1/2}\left(1 - \frac{3GM}{Rc}\right)^{-1/2}$$

Radial Motion ($L=0$)

$$u^\mu = \left(\frac{dt}{d\tau}, \frac{dr}{d\tau}, 0, 0\right)$$

$$L=0 \quad \Rightarrow \quad \frac{1}{2}\left(\frac{dr}{d\tau}\right)^2 + \frac{1}{2} - \frac{GM}{r} = \frac{1}{2}\mathcal{E}^2$$

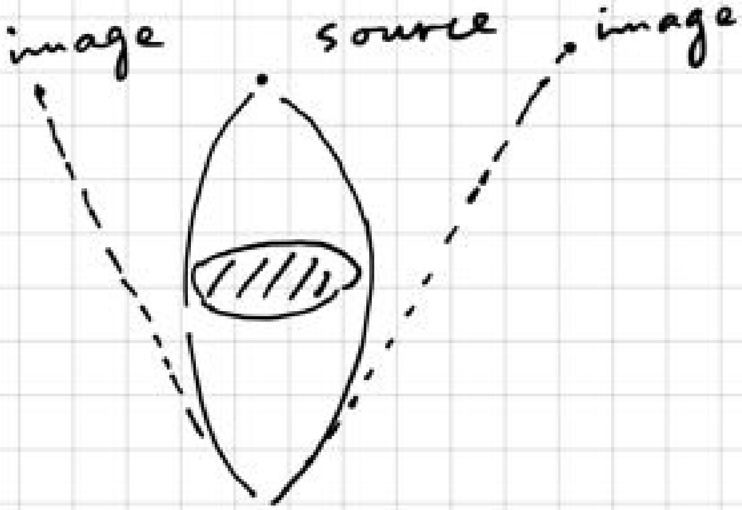
Let $\mathcal{E}=1$ freed from ∞.

$$\frac{dr}{d\tau} = \pm\sqrt{\frac{R_s}{r}}$$

Some missing part...

Relativistic astrophysics

1) gravitational lensing

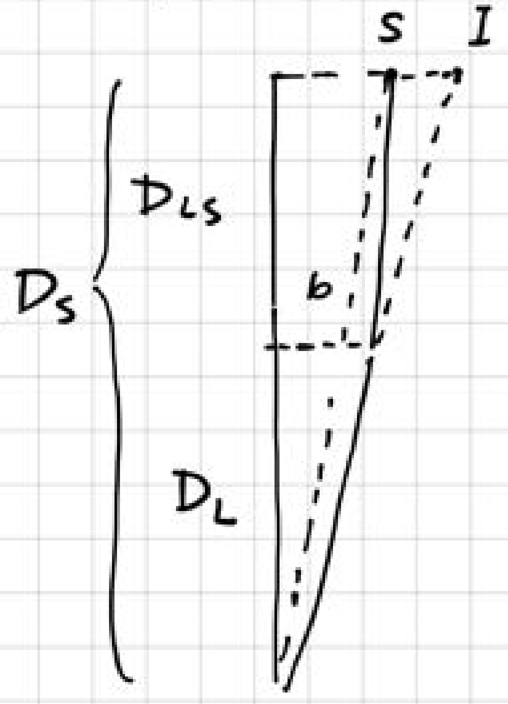


Thin lense approximation.

i) light goes straightly.

ii) deflation only at location of length

iii) lense & source as points.



$$D_S \theta_I = D_S \theta_S + D_{LS} \cdot \alpha$$

$$D_S \theta_I = D_S \theta_S + D_{LS} \cdot \frac{2R_S}{b}.$$

$$D_S \theta_I = D_S \theta_S + D_{LS} \frac{2R_S}{\theta_I D_L}$$

$$\theta_I = \theta_S + \frac{2R_S D_{LS}}{D_S D_L} \frac{1}{\theta_I}$$

$$\equiv \theta_S + \frac{\theta_E^2}{\theta_I} \quad \left(\theta_E^2 \equiv \frac{2R_S D_{LS}}{D_S D_L} \right)$$

if $\theta_S = 0$ $\theta_I = \theta_E$ → Einstein angle (radius)

if $\theta_S \neq 0$ $\theta_I = \frac{-\theta_S \pm \sqrt{\theta_S^2 + 4\theta_E^2}}{2}$

θ_E → characteristic angular scale

distance between stars ~ pc

galaxy radius ~ kpc.

distance between galaxies ~ Mpc.

i) Microlensing.

$M_L \sim M_\odot \Rightarrow R_S \sim 1$ km.

$D_L . D_S . D_{LS} \sim$ kpc.

$\theta_E \sim (10^{-3})''$ ⟶ luminosity.

to detect MACHO

massive astrophysical compact halo object.

exoplanets.

ii) Macrolensing. (strong lensing).

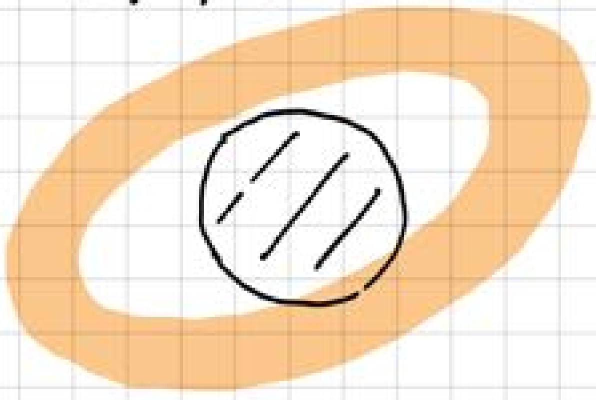
$M_L \sim 10^{11} M_\odot$ $D_L . D_S . D_{LS} \sim 1$ Mpc.

$\theta_E \sim 1''$

iii) weak lensing

cosmic effect.

2) Accretion disk



Fe Kα 6.4 keV.

$\downarrow$

extremely broadened

$\begin{cases} \text{gravitational redshift} \\ \text{Doppler effect.} \end{cases}$

$i = \frac{\pi}{2}$. circular motion.

Schwarzchild spacetime

$$ds^2 = -\left(1 - \frac{R_s}{r}\right) dt^2 + \left(1 - \frac{R_s}{r}\right)^{-1} dr^2 + r^2 d\Omega^2$$

$$u_s = \frac{dt}{d\tau} (1\ 0\ 0\ \Omega) , \quad \Omega = \left(\frac{GM}{r^3}\right)^{1/2}$$

$$\frac{dt}{d\tau} = \left(1 - \frac{3GM}{r}\right)^{-1/2}$$

$\hat{\xi} = \partial t \quad \hat{\eta} = \partial \phi$

$\xi^\mu = (1\ 0\ 0\ 0) \quad \eta^\mu = (0\ 0\ 0\ 1)$

$$\hat{u}_s = \frac{dt}{d\tau}(\hat{\xi} + \Omega \hat{\eta}) \qquad \begin{cases} E = \hat{\xi}\cdot\hat{p} \\ L = \hat{\eta}\cdot\hat{p} \end{cases} \quad b^2 = \frac{L^2}{E^2}$$

① static observer at infinity.

$\hat{u}_0 = \hat{\xi}$

② photon ~ null geodesic

$E_* = -\hat{u}_s \cdot \hat{p}$

$E_R = -\hat{u}_R \cdot \hat{p}$

$$\Rightarrow \frac{\omega_R}{\omega_*} = \frac{E_R}{\omega_*}$$

(gravitational redshift & transverse doppler)

i) $\theta = 0$ or π. $b = 0$.

redshift

$$\frac{\omega_R}{\omega_*} = \frac{E}{u^t(E + \Omega L)} \overset{L=0}{=\!=} \left(1 - \frac{3GM}{r}\right)^{1/2} < 1$$

(ii) $\theta = \pm\frac{\pi}{2}$ in star frame. $p^r = 0$. $p^\theta = 0$

$$\hat{p}\cdot\hat{p} = 0 \Rightarrow -\left(1-\frac{R_s}{r}\right)^{-1}(p^t)^2 + r^2\sin^2\theta\,(p^\phi)^2 = 0$$

$$b = \frac{L}{E} = \frac{r^2 p^\phi}{\left(1-\frac{R_s}{r}\right)p^t}$$

$$b = r\left(1-\frac{R_s}{r}\right)^{-1/2}$$

$$\therefore\ \frac{\omega_R}{\omega_*} = \frac{1}{u^t(1+\Omega|b|)}$$

$$= \frac{1}{u^t(1\pm\Omega b)}$$

$$= \left(1-\frac{3GM}{r}\right)^{1/2}\left[1\mp\left(\frac{r}{GM}-2\right)^{-1/2}\right]^{-1}$$

$$\therefore\ \frac{GM}{r} \ll 1$$

$$\frac{\omega_R}{\omega_*} = \left(1-\frac{3GM}{2r}+\cdots\right)\left(1\mp\underbrace{\sqrt{\frac{GM}{r}}}_{\text{doppler}}+\frac{GM}{r}+\cdots\right)$$

$$= 1\mp\sqrt{\frac{GM}{r}} - \frac{GM}{2r} = 1+v+\frac{1}{2}v^2-\frac{GM}{r} \leftarrow \text{GR.}$$

$r = 6GM$

i) $\theta = 0$ or π $\quad \frac{\omega}{\omega_*} = \frac{1}{\sqrt{2}}$.

ii) $\theta = \pm\frac{\pi}{2}$. $\quad \frac{\omega}{\omega_*} = \frac{1}{\sqrt{2}}\left(1 \pm \frac{1}{2}\right)^{-1}$

$$= \sqrt{2} \text{ or } \frac{\sqrt{2}}{3}$$

blueshift from Doppler motion

3) Binary system

· pulsar.

GW radiation.

PSR B1913+16.

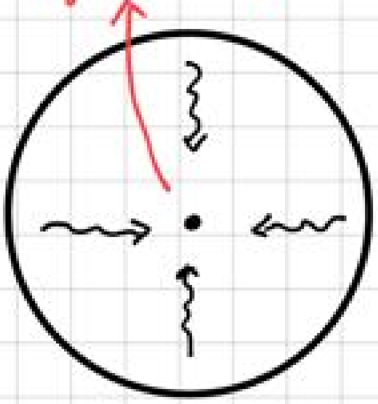
orbital decay

$$ds^2 = -\left(1-\frac{R_s}{r}\right)dt^2 + \left(1-\frac{R_s}{r}\right)dr^2 + r^2 d\Omega^2$$

$\Rightarrow$ **Schwarzchild BH**

Why a vaccum Einstein eqs can be used to describe a BH?

singularity.



spherically collapse.

$\downarrow$

form a singularity.

BH in Newtonian gravity. (Black Star)

escape velocity. $v = \sqrt{\frac{2GM}{r}}$

if $v > c \Rightarrow r < \frac{2GM}{c^2} = R_s$.

$\rho \to \frac{1}{M^2}$

use Schwarzchild metric to describe

a BH.

Event Horizon/

$r=0$ singularity. ↑ infinite redshift.

$r=R_s$ coordinate singularity. $\frac{\lambda_R}{\lambda_e} = \left(\frac{1-\frac{R_s}{R_R}}{1-\frac{R_s}{R_e}} \right)^{\frac{1}{2}}$

Massive particle free-falling.

$\Delta\tau \big|_{r_0 \to R_s}$ finite.

observer at ∞.

$\Delta t \big|_{r_0 \to R_s}$ infinite.

↳ a illusion!

Some problems in using Schwarzchild

coordinate. into BH

Event horizon → killing horizon

∂_t.

$r > R_s$. $\xi = \partial_t$ timelike

$r < R_s$ $\hat{\xi} = \partial_t$ spacelike.

$\Rightarrow$ $r > R_s$ t "time"

r "space"

$r < R_s$ t "space"

r "time"

$r < R_s$. massive particle

$d\tau^2 = -ds^2 < 0$

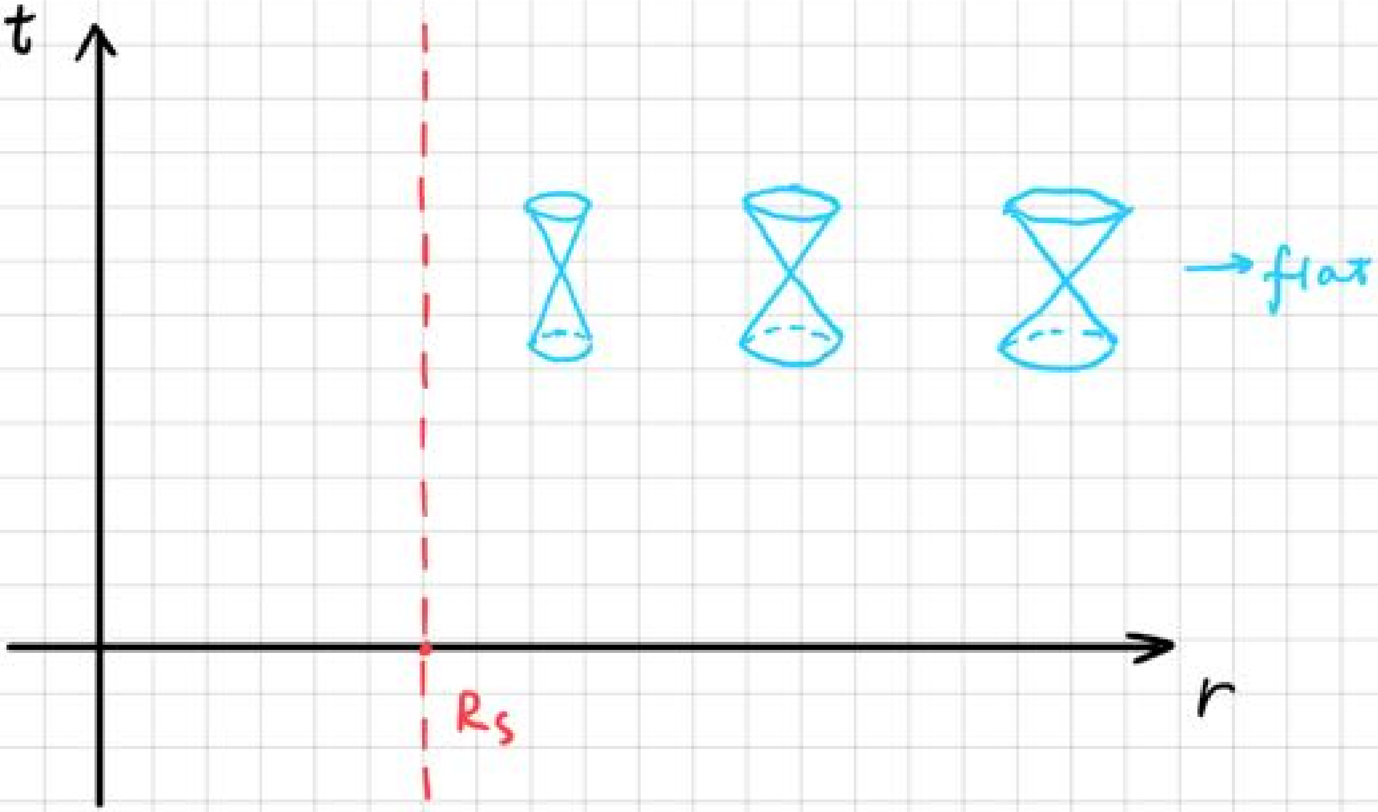
if $dr = 0 \Rightarrow d\tau^2 > 0$.

$\Rightarrow$ cannot find $r = \text{const}$ when $r < R_s$.

Light cone structure.

null curve $ds^2 = 0$.

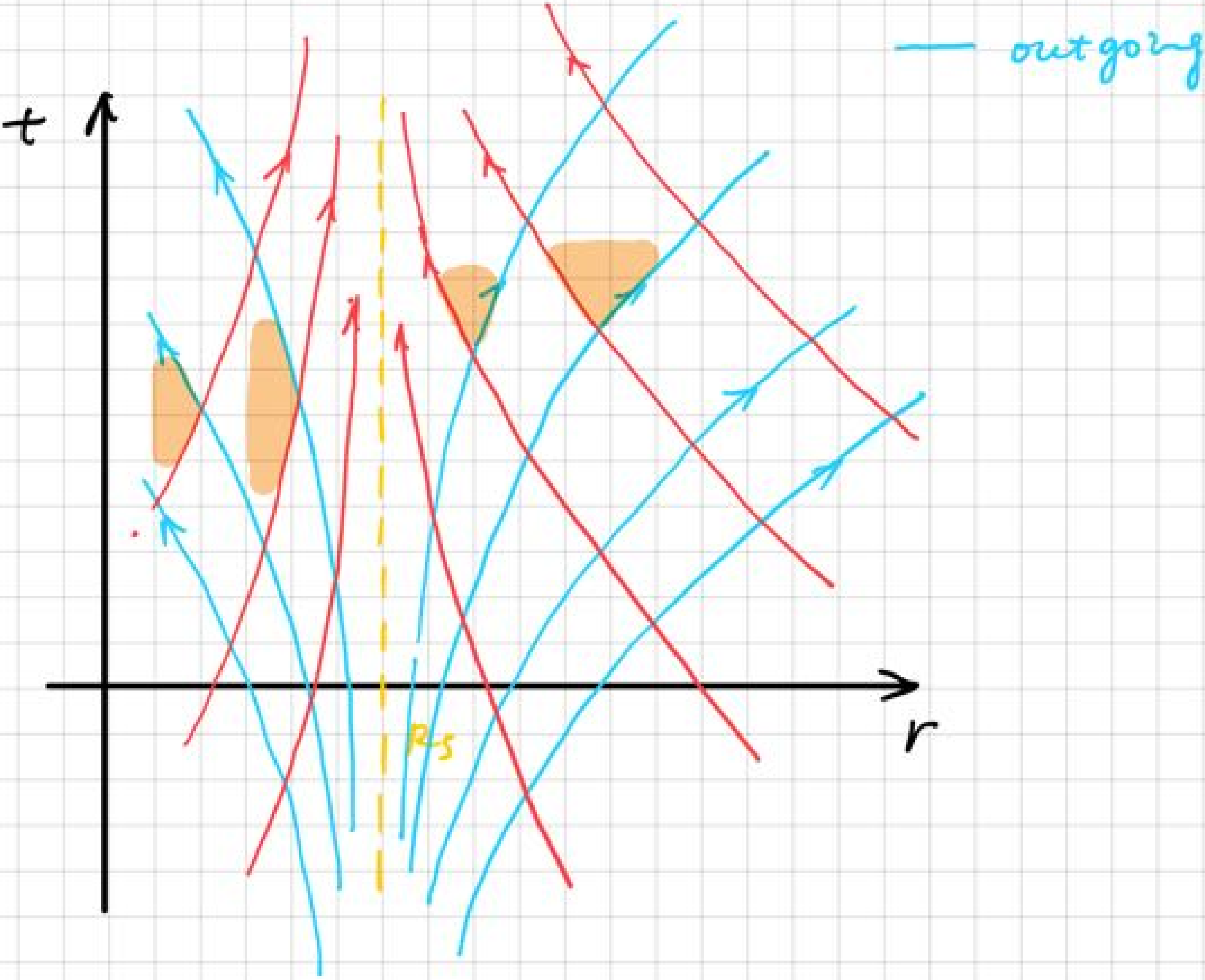
$$\frac{dt}{dr} = \pm \left(1 - \frac{R_s}{r}\right)^{-1}$$



$$\begin{cases} t = r + R_s \ln \left| \dfrac{r}{R_s} - 1 \right| + C & \text{outgoing.} \\ t = -r - R_s \ln \left| \dfrac{r}{R_s} - 1 \right| + C & \text{infalling.} \end{cases}$$

↓

works even at $r < R_s$.



we choose "tortoise coordinate"

$$r_* = r + R_s \ln \left| \frac{r}{R_s} - 1 \right|$$

$r_* \to -\infty \quad r_* \in (-\infty, \infty)$.

$$\Rightarrow ds^2 = -\left(1 - \frac{R_s}{r}\right)\left(-dt^2 + dr_*^2\right) + r^2 d\Omega^2.$$

null curve: $\frac{dt}{dr_*} = \pm 1$

Another idea: a family of null radial geodesic.

we use the integration constant as a coordinate.

$$\begin{cases} t = r + R_s \ln \left| \frac{r}{R_s} - 1 \right| + C_1 & \text{outgoing.} \\ t = -r - R_s \ln \left| \frac{r}{R_s} - 1 \right| + C_2 & \text{infalling.} \end{cases}$$

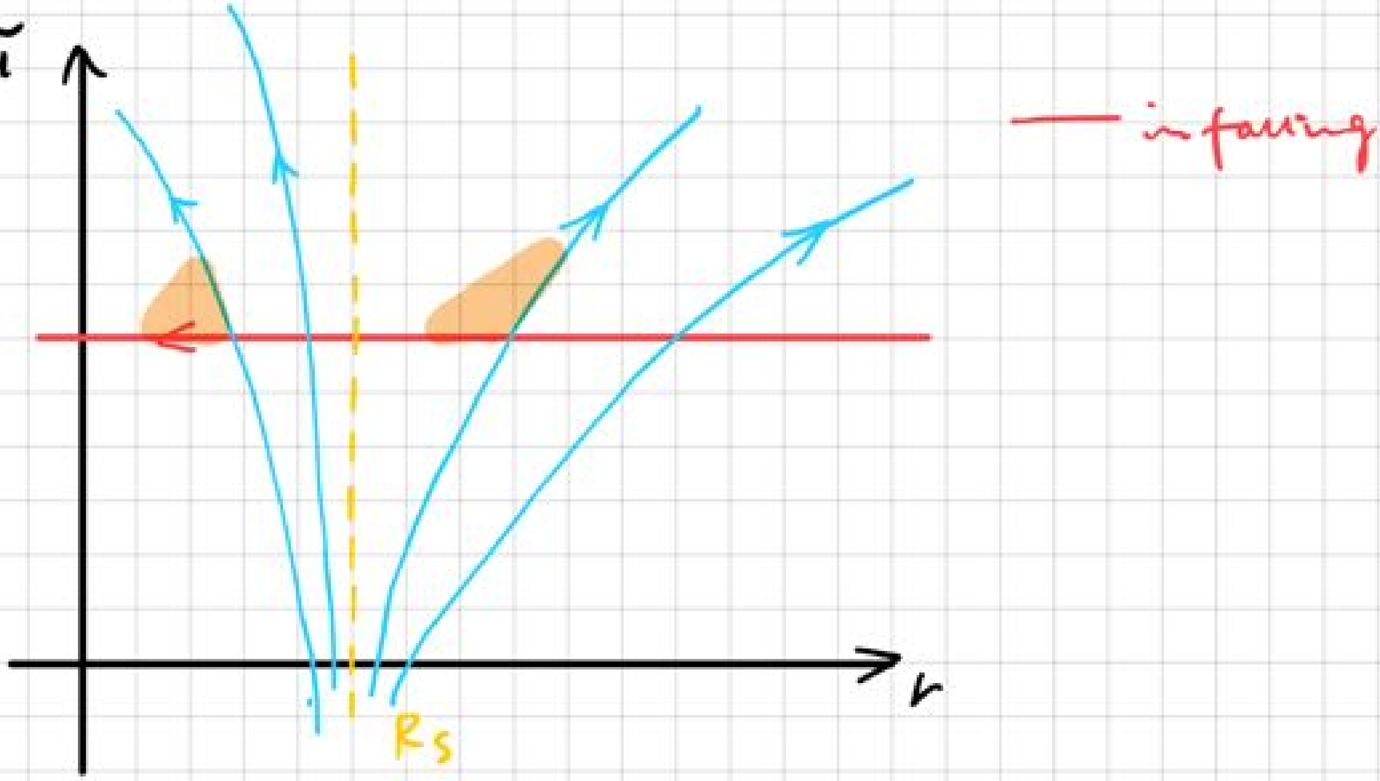
Def: $\begin{cases} \tilde{u} = t + r_* & \text{infalling} \\ \tilde{v} = t - r_* & \text{outgoing.} \end{cases}$

i) $(\tilde{u}, r, \theta, \phi)$ advanced Eddington-Finkelstein coordinate

$$\Rightarrow ds^2 = -\left(1 - \frac{R_s}{r}\right) d\tilde{u}^2 + (dr\, d\tilde{u} + d\tilde{u}\, dr) + r^2 d\Omega^2$$

$$\frac{d\tilde{u}}{dr} = \begin{cases} 0 & \text{infalling} \\ 2\left(1 - \frac{R_s}{r}\right)^{-1} & \text{outgoing} \end{cases}$$

still singular at horizon



we can keep refining it.

$$ds^2 = -\left(1-\frac{R_s}{r}\right) d\tilde{u}^2 + (dr\,d\tilde{u} + d\tilde{u}\,dr) + r^2 d\Omega^2$$

let $t' = \tilde{u} - r = t + R_s \ln\left|\frac{r}{R_s} - 1\right|$

$$ds^2 = -\left(1-\frac{R_s}{r}\right) dt'^2 + \frac{2R_s}{r}\, dt'\, dr + \left(1-\frac{R_s}{r}\right) dr^2 + r^2 d\Omega^2$$

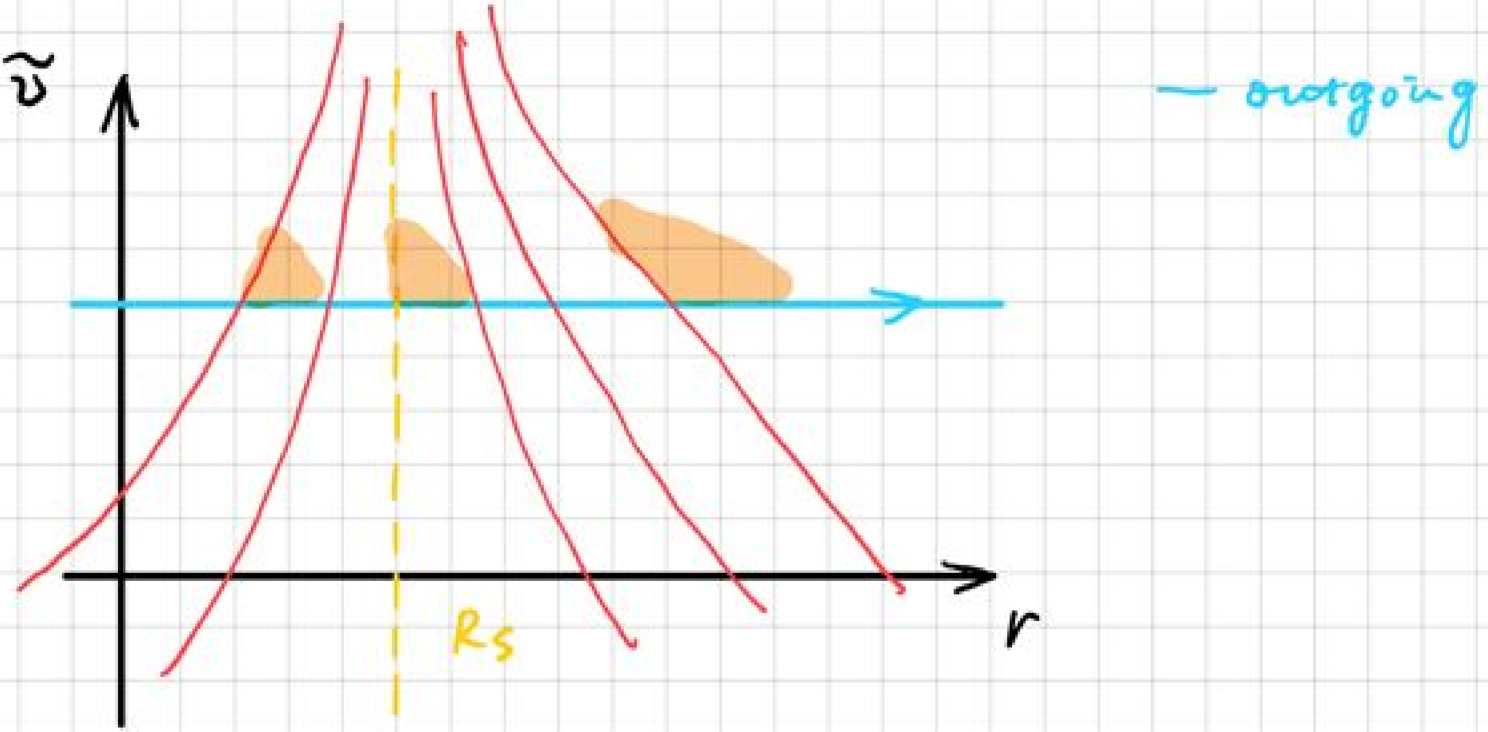
infalling $\Rightarrow$ $t' + r = \text{const}$.

- lost of time reversal invariability.

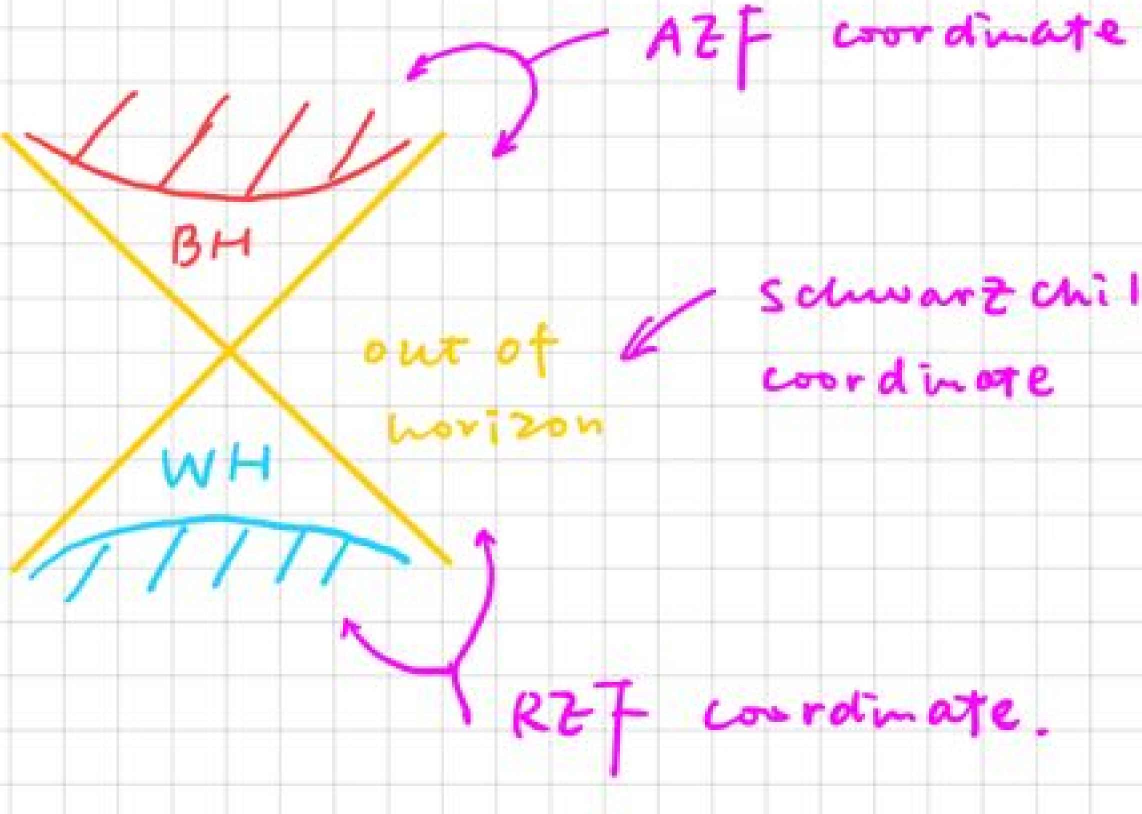
2) $(\tilde{v}, r, \theta, \phi)$ Retarded EF coordinate

$$ds^2 = -\left(1 - \frac{R_s}{r}\right) d\tilde{v}^2 - \left(d\tilde{v}\, dr + dr\, d\tilde{v}\right) + r^2 d\Omega^2$$

$$\frac{d\tilde{v}}{dr} = \begin{cases} 0 & \text{outgoing.} \\ -2\left(1 - \frac{R_s}{r}\right)^{-1} & \text{infalling.} \end{cases}$$



time reversal of BH → white hole



How do we know the spacetime is maximum?

$\forall p \in M$.

$\{x^\mu(p)\}$ extended

if $\begin{cases} \text{reach } \infty \nearrow \text{geodesic completeness.} \\ \text{meet singularity.} \end{cases}$

Step 1: we choose $(\tilde{u}, \tilde{v}, \theta, \phi)$

$$ds^2 = \frac{1}{2}\left(1 - \frac{R_s}{r}\right)(d\tilde{u}\,d\tilde{v} + d\tilde{v}\,d\tilde{u}) + r^2 d\Omega^2$$

$$\frac{1}{2}(\tilde{u} - \tilde{v}) = r + R_s \ln\left(\frac{r}{R_s} - 1\right)$$

conformal flat.

Step 2: $u' = e^{\tilde{u}/2R_s}$ $\qquad v' = e^{-\tilde{v}/2R_s}$.

$r > R_s$:

$$u' = \left(\frac{r}{R_s} - 1\right)^{1/2} e^{\frac{r+t}{2R_s}}$$

$$v' = \left(\frac{r}{R_s} - 1\right)^{1/2} e^{\frac{r-t}{2R_s}}$$

$$ds^2 = \frac{2R_s^3}{r} e^{-\frac{r}{R_s}} \left(du'dv' + dv'du' \right) + r^2 d\Omega^2$$

Step 3:

$$\begin{cases} u = \frac{1}{2}(u' - v') \\ v = \frac{1}{2}(u + v') \end{cases}$$

$$ds^2 = \frac{4R_s^2}{r} e^{-\frac{r}{R_s}} \left(-dv^2 + du^2 \right) + r^2 d\Omega^2$$

$$u'^2 - v'^2 = \left(\frac{r}{R_s} - 1 \right)^{r/R_s}.$$

$(v\ u\ \theta\ \phi)$: Kruskal-Szekeres coordinate

i) no coordinate singularity.

ii) $\begin{cases} v \sim t \\ u \sim x \end{cases}$ fixed θ, ϕ: $v = \pm u + \text{const}.$

$\Downarrow$

conformally flat with

2D Minkowski space.

(same light cone).

ii) Horizon $r = R_s$

$\Rightarrow u^2 - v^2 = 0 \Rightarrow u = \pm v$.

iii) $r = \text{const}$, $u^2 - v^2 = \text{const}$

$r > R_s \quad u^2 - v^2 > 0$

$r < R_s \quad u^2 - v^2 < 0$.

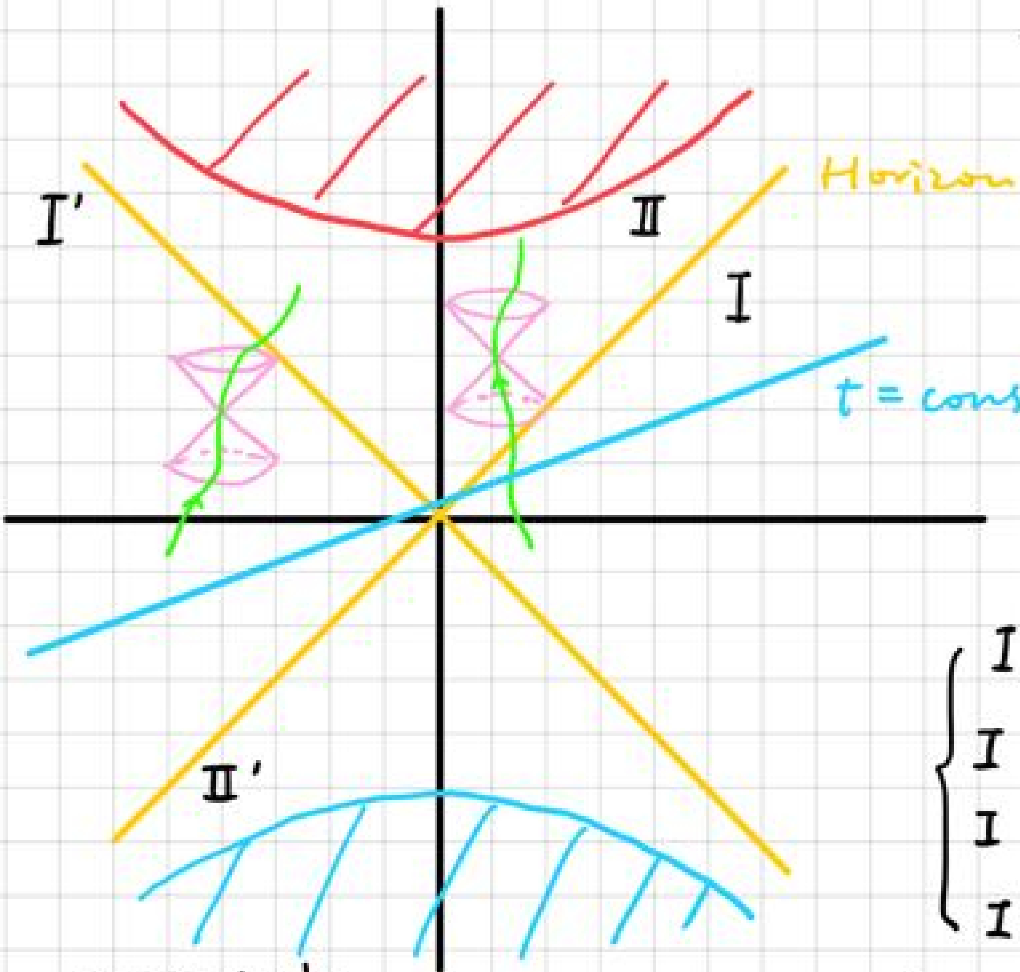
$r = 0 \quad u^2 - v^2 = -1$

why we don't consider $r<0$.

$r<0$, $M \to -M$ naked singularity ×

$t = \text{const}$

$\Rightarrow \frac{v}{u} = \tanh\frac{t}{2R_s}$



I Schwarzchild
I & II AEF
I & II' REF
I' a newly found asymtopic flat space.

$\mathrm{I} \xleftrightarrow{\text{worm hole}} \mathrm{I}'$

Why not we define a coordinate from a massive particle in radial motion?

Painleve coordinate

$$u^\mu = \left(\left(1 - \frac{R_s}{r}\right)^{-1} \ -\sqrt{\frac{R_s}{r}} \quad 0 \quad 0 \right)$$

$$u_\mu = \left(-1 \, , \, -\sqrt{\frac{R_s}{r}} \left(1 - \frac{R_s}{r}\right)^{-1} \quad 0 \quad 0 \right)$$

find $T(t,r)$ s.t $dT \propto u_\mu dx^\mu$.

$$\Rightarrow T = t + \int^r \left(1 - \frac{R_s}{r}\right)^{-1} \sqrt{\frac{R_s}{r'}} \, dr'$$

use T as time coordinate

$$dT = dt + \left(1 - \frac{R_s}{r}\right) \sqrt{\frac{R_s}{r}} \, dr.$$

$$\Rightarrow ds^2 = -dT^2 + \left(dr + \sqrt{\frac{R_s}{r}} \, dT \right)^2 + r^2 d\Omega^2$$

Tetrad $\tilde{\theta}^0 = dT$ $\quad \tilde{\theta}^1 = dr + \sqrt{\frac{R_s}{r}}\, dr$

$$ds^2 = \eta_{nm}\,\tilde{\theta}^m \tilde{\theta}^n$$

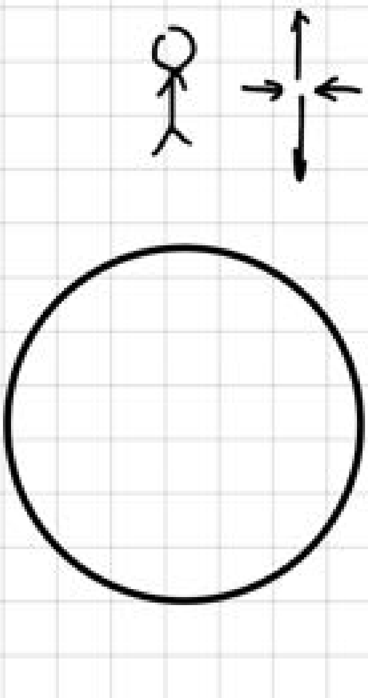
$$\tilde{\theta}^0 = e^m{}_\mu\, dx^\mu$$

Geodesic deviation.

$$\frac{D^2 \eta^m}{D\tau^2} = R^m{}_{00n}\, \eta^n$$

$$\Rightarrow \begin{cases} \dfrac{D^2\eta^1}{D\tau^2} = \dfrac{R_s}{r^3}\,\eta^1 \\[2ex] \dfrac{D^2\eta^2}{D\tau^2} = -\dfrac{R_s}{2r^3}\,\eta^2 \end{cases}$$

$\to$ tidal force.



I' another asymptotic flat space in Kruskal coordinate.

Can we connect I & I' ? (by some space-like hyper-surface)

$t=0$ or $v=0$, $\theta=\frac{\pi}{2}$

$\downarrow$

$$ds^2 = \frac{4R_s^3}{r} e^{-\frac{r}{R_s}} \left(-dv^2+du^2\right) + r^2 d\Omega^2$$

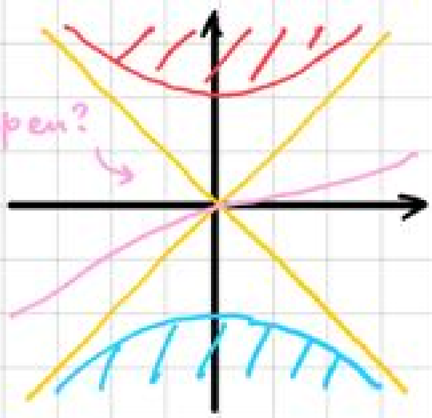
$$\Rightarrow ds^2 = \left(1-\frac{R_s}{r}\right)^{-1} dr^2 + r^2 d\phi^2$$

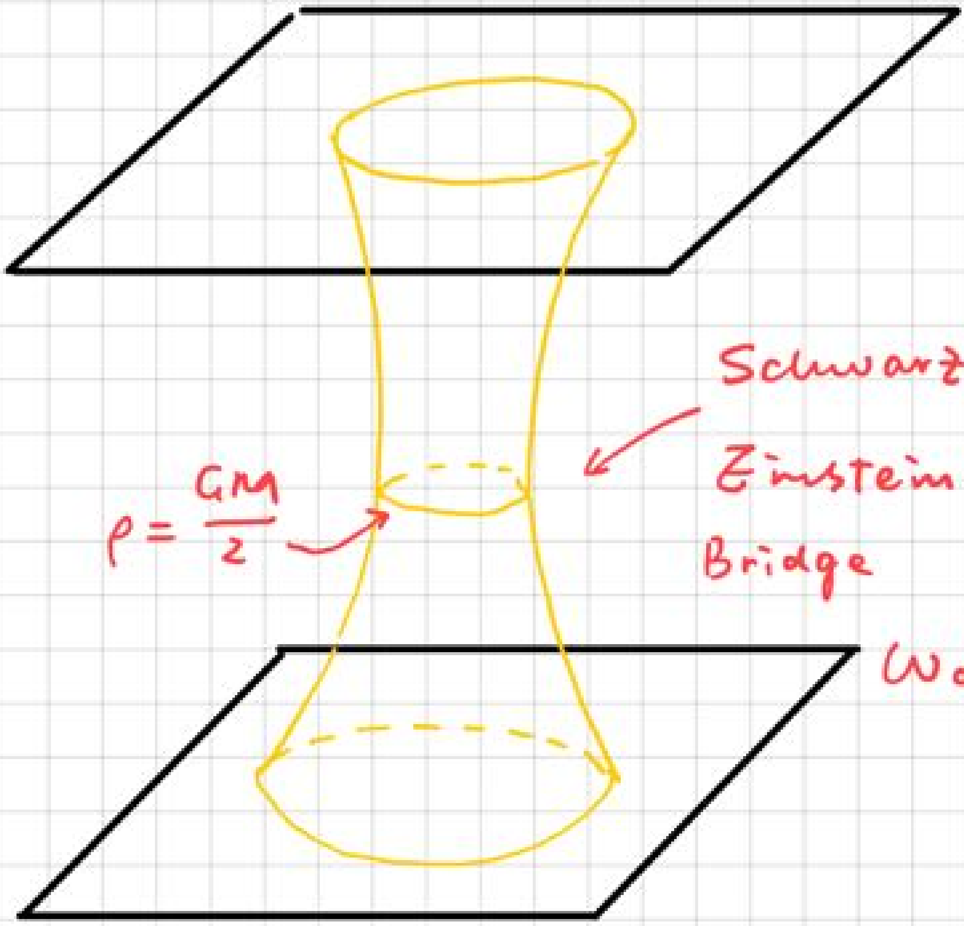
We want to find a induced metric to show it in 3D.

$(r, \phi) \rightarrow (z, r, \phi)$. $ds^2 = dz^2 + r^2 d\phi^2$

$$\Rightarrow \left(\frac{dz}{dr}\right)^2 + 1 = \left(1-\frac{R_s}{r}\right)^{-1}$$

$$z(r) = \sqrt{4R_s(r-R_s)} + C. \quad (r \geq R_s)$$





Isotropic coordinates. (only works at I' & I)

$$r = \left(1 + \frac{GM}{2\rho}\right)^2 \rho \rightarrow r \geq R_s$$

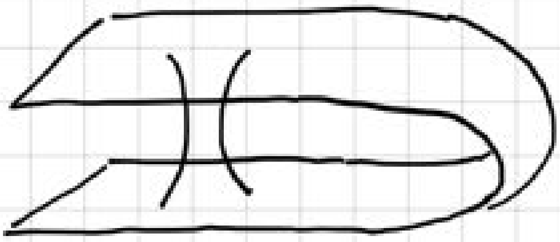
$\rightarrow (t, \rho, \theta, \phi)$

$$\Rightarrow ds^2 = -\left(\frac{1 - \frac{GM}{2\rho}}{1 + \frac{GM}{2\rho}}\right)^2 dt^2 + \left(1 + \frac{GM}{2\rho}\right)^4 \underbrace{\left(d\rho^2 + \rho^2 d\Omega^2\right)}_{\text{conformally flat to } \mathbb{R}^3}$$

$\rho \rightarrow \frac{(GM)^2}{4\rho}$ $\quad ds^2$ inv.

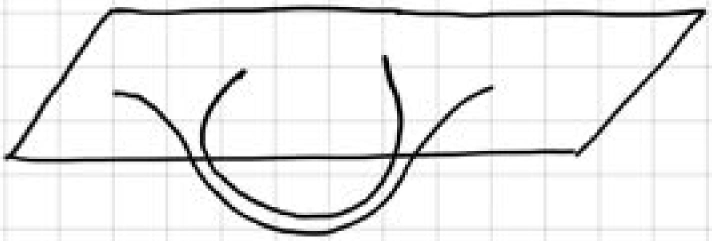
$\Rightarrow$ t fixed, 2 different ρ, except $\rho = \frac{GM}{2}$

Einstein-Rosen bridge (1935)



X not possible
space-like curve

Whealer "Geon" (1955)



"Transversable" Worm Hole?

{ "time" infinite
{ No event horizon → no singularity

1988 Moris-Throne

↓

violate null-energy condition.
(should not be classical material)

EPR = ER? Susskind & Maldacena

BH formation

Stellar mass BH ~ tens of $M_\odot$

Super massive BH $> 10^6 M_\odot$

merge?

tiny BH?

primodial BH?

WD $1.44 - 1.46 M_\odot$ Chandrasekhar limit

$M > 8 M_\odot$, continue collapse. SN

→ reverse β - decay

$$e^- + p \longrightarrow n + \nu_e^-$$

↓

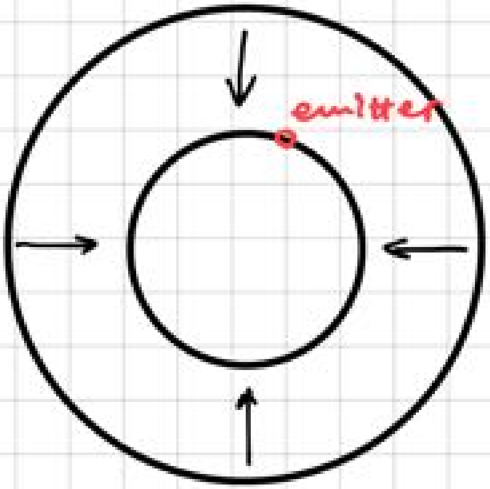
neutron star.

$M > 20 M_\odot$ BH. (M, J, B)

Singularity Theorem

Penrose & Hawking.

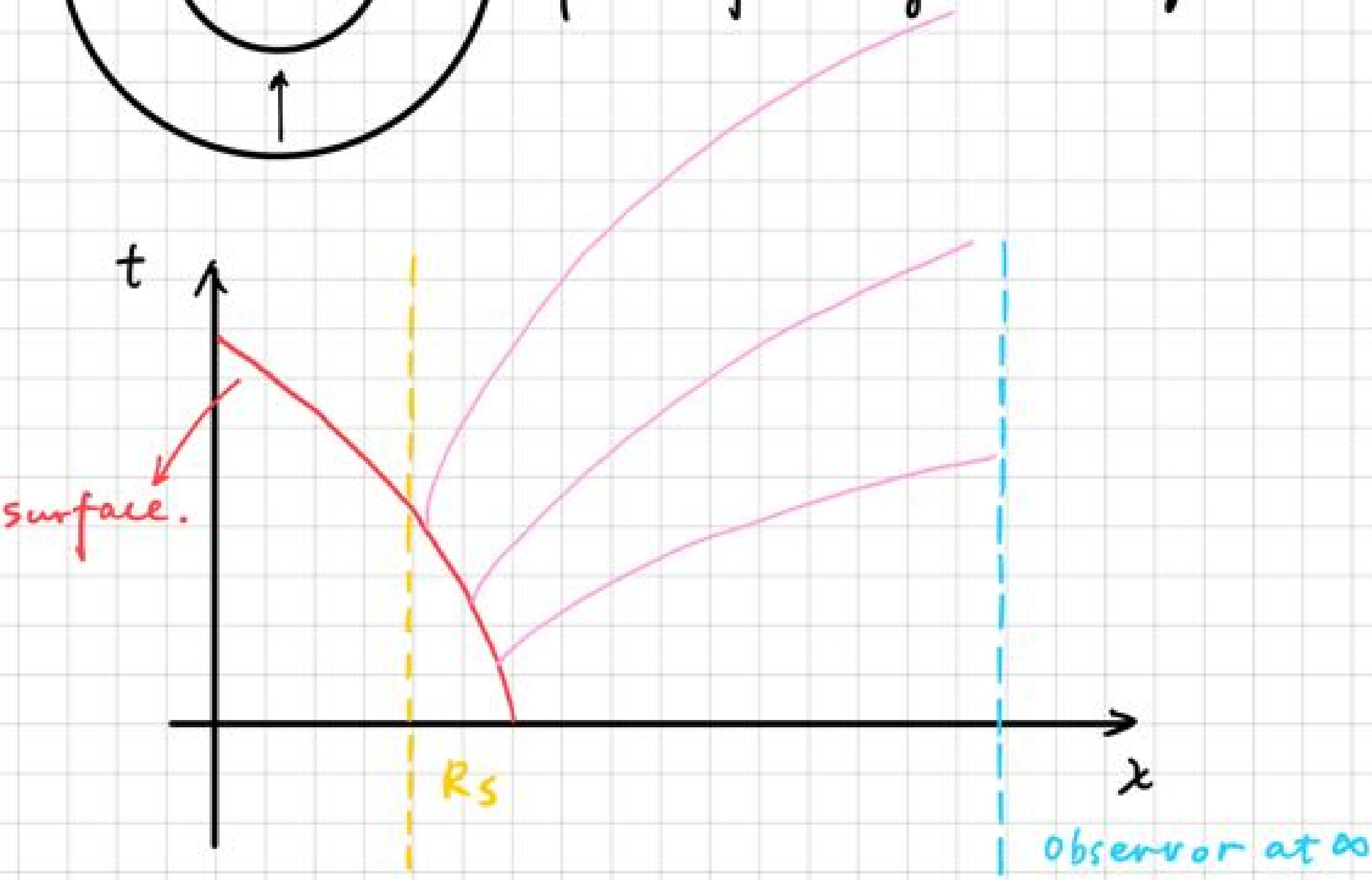
Collapse starts → BH.



Spherically symmetric

$p<0$.

free falling radially.



$$r_E(t_R) = R_S + a\exp\left(-\frac{t_R}{2R_S}\right).$$

typical timescale for observor $\sim \frac{2R_S}{c}$

$\sim 10^{-6}\left(\frac{M}{M_\odot}\right)$ s

Redshift $\quad \dfrac{\nu_R}{\nu_Z} = \dfrac{u^\mu{}_R \, p_\mu(R)}{u^\mu{}_Z \, p_\mu(Z)}$

$$Z = 1, \quad u_Z^\mu = \left(\left(1 - \frac{R_s}{r}\right)^{-1}, \ -\left(\frac{R_s}{r}\right)^{1/2}, \ 0, 0 \right)$$

$$u_R^\mu = (1, 0, 0, 0)$$

$$\hat{p} \cdot \hat{p} = 0. \Rightarrow \quad p_1 = -\left(1 - \frac{R_s}{r}\right)^{-1} p_0.$$

$$\therefore \frac{\nu_R}{\nu_Z} = \frac{p_0(R)}{u_Z^0 p_0(Z) + u_Z^1 p_1(Z)}$$

$$= \left(u^0_Z - \left(1 - \frac{R_s}{r}\right)^{-1} u_Z^1 \right)^{-1}$$

$$= \left(1 - \frac{R_s}{r}\right)\left(1 + \sqrt{\frac{R_s}{r}}\right)^{-1}$$

$$= 1 - \sqrt{\frac{R_s}{r}} \quad \sim e^{-t/2R_s}$$

Luminosity $\sim e^{-t/R_s}$

Vaccum Einstein eq.

a Star

What eq describes here?

Schwarzchild eq. (external)

$G_{\mu\nu} = 8 T_{\mu\nu}$ at $r=R$ → Schwarzchild

$$ds^2 = -e^{2\alpha}dt^2 + e^{2\beta}dr^2 + r^2 d\Omega^2$$

$$T_{\mu\nu} = (p+\rho)u_\mu u_\nu + p\, g_{\mu\nu}$$

$$\begin{cases} G_{ti} = 0 \\ g_{ti} = 0 \end{cases}$$

$$\Downarrow$$

$$T_{ti} = 0.$$

$$\Downarrow$$

$\hat{u}\cdot\hat{u} = -1 \Rightarrow u_0 \neq 0$

$$(p+\rho)u_0 u_i = 0.$$

$$\Downarrow$$

$$u_i = 0$$

$$\Downarrow$$

$$g_{00}{u^0}^2 = -1 \Rightarrow u_0 = e^{\alpha} \Rightarrow \begin{cases} T_{00} = \rho e^{2\alpha} \\ T_{rr} = p g_{rr} = p e^{2\beta} \\ T_{\theta\theta} = p r^2 \\ T_{\phi\phi} = p r^2 \sin^2\theta \end{cases}$$

Independent Eqs.

G_{tt} $$\frac{e^{-2\beta}}{r^2}\left(2r\partial_r\beta - 1 + e^{2\beta}\right) = 8\pi G\rho(r)$$

G_{rr} $$\frac{e^{-2\beta}}{r^2}\left(2r\partial_r\alpha + 1 - e^{2\beta}\right) = 8\pi G p(r)$$

$G_{\theta\theta}$

$$e^{2\beta} = \left(1 - \frac{Gm(r)}{r}\right)^{-1}$$ at $r=R \rightarrow$ Sch

$$\Rightarrow \frac{dm(r)}{dr} = 4\pi r^2\rho(r)$$ looks like in a flat space!

$$\Rightarrow m(r) = \int_0^r 4\pi r'^2 \rho(r')\,dr'$$

$m(R) = M_{ADM}$. (the "mass" feel at infinity, contains potential energy)

actually the proper volume element

$$\overline{M}(r) = \int_0^r e^{\beta r'} 4\pi r'^2 \rho(r')\,dr'$$

$$= \int_0^r \left(1 - \frac{Gm(r')}{r}\right)^{-1/2} 4\pi r'^2 \rho(r')\,dr'$$

$$m(r) = m_0(r) + \upsilon(r) + \Omega(r)$$

$$m_0(r) = \int_0^r (\mu_i n)\sqrt{\gamma}\, d^3r \quad \text{rest mass.}$$

$$\upsilon(r) = \int_0^r (\rho - \mu_i n)\sqrt{\gamma}\, d^3r$$

$$\Omega(r) = -\int_0^r \rho'(r) \left(\left(1 - \frac{2Gm(r')}{r}\right)^{-1/2} - 1 \right) 4\pi r'^2 dr'$$

Ideal case: all of Baryon:

$$B = \int_0^R 4\pi r'^2 e^{\beta(r')} n(r')\, dr'$$

$$\overline{M} = m_N B.$$

if $n(r) = n$, $\rho(r) = \rho^*$

$$m(r) = \frac{4\pi}{3}\rho_* r^3, \quad B = \frac{4\pi}{3} n R^3 f(x)$$

$$f(x) = \frac{3}{2}\,\frac{\sin^{-1}x - x\sqrt{1-x^2}}{x^3}$$

$$x = R\sqrt{A}$$

$$A = \frac{8\pi G}{3}\rho_*$$

$x \ll 1 \quad f(x) = 1 + \frac{3}{10} x^2$

$$f(x) - 1 = \frac{3}{5} \frac{GM}{rc^2}$$

binding energy in Newtonian potential

G_{rr} eq

$$(\rho + p) \frac{d\alpha}{dr} = - \frac{dp}{dr} \Leftarrow \nabla^\mu T_{\mu\nu} = 0.$$

$\Downarrow$

pressure change pressure gradient (relativistic effect)

$$\left. \begin{aligned} \frac{dp}{dr} &= - \frac{(\rho + p)[Gm(r) + 4\pi r^3 p(r)]}{r(r - 2Gm(r))} \\ \frac{dm}{dr} &= 4\pi r^2 \rho(r) \end{aligned} \right\}$$

Tolman – Oppenheimer – Volkoff (TOV eq)

+ EOS ? ($p = p(\rho)$) $\Rightarrow$ spacetime in a star (non-rotating)

Boundary condition:

$$\begin{cases} ds^2|_{r=R} = ds^2_{Sch}(M) \\ p(0) = ? \quad p(R) = 0 \\ m(0) = 0 \quad m(R) = M \end{cases}$$

$$\frac{dp}{dr} < 0.$$

Example:

$$\rho(r) = \begin{cases} \rho_* & (r \le R) \\ 0 & (r > R) \end{cases}$$

(good approximation to neutron star)

$$m(r) = \frac{4}{3}\pi r^3 \rho_*$$

$$\frac{dp}{dr} = -\frac{4\pi}{3} G r \frac{(\rho_* + p)(\rho_* + 3p)}{1 - \frac{8\pi}{3} G \rho_* r^2}$$

$\Rightarrow$

$$\frac{\rho_* + 3p}{\rho_* + p} = \frac{\rho_* + 3p_0}{\rho_* + p_0}\left(1 - \frac{8\pi G}{3}\rho_* r^2\right)^{1/2}$$

$$\Rightarrow p_0 = \rho_* \frac{1 - \left(r - \frac{R_S}{R}\right)^{1/2}}{3\left(1 - \frac{R_S}{R}\right)^{1/2} - 1}$$

$R = \frac{9}{4} GM \quad p_0 \to \infty \quad$ Collapse.

Buchdahl's Theorem:

There's no stable star with $R < \frac{9}{4} GM$

Reissner - Nordström spacetime

1) $G_{\mu\nu} = 8\pi G T_{\mu\nu}$

$$S = \int \sqrt{-g}\, d^4x \left(R - \frac{e^2}{4} F^{\mu\nu} F_{\mu\nu} \right)$$

2) $\begin{cases} dF = 0 & (F = dA) \\ d{*}F = 0 & (\text{source free}) \end{cases}$

Similarly. we have

$$ds^2 = -e^{2\alpha(r)} dt^2 + e^{2\beta(r)} dr^2 + r^2 d\Omega^2$$

electric field   magnetic field

$$F_{\mu\nu}(r) = F_{tr}(r)\, dt \wedge dr + F_{\theta\varphi}(\theta)\, d\theta \wedge d\varphi$$

(satisfy spherical symmetry)

$$\Rightarrow \begin{cases} F_{tr} = \dfrac{Q}{4\pi} \dfrac{e^{\alpha+\beta}}{r^2} \longrightarrow \vec{E} \\ F_{\theta\varphi} = \dfrac{P}{4\pi} \sin\theta \longrightarrow \vec{B} \end{cases}$$

$+ \; G_{\mu\nu} = 8\pi G T_{\mu\nu}$

$\Downarrow$

$\alpha + \beta = 0$ & $ds^2_{RN} = -\Delta dt^2 + \Delta^{-1} dr^2 + r^2 d\Omega^2$

$$\Delta = 1 - \frac{2GM}{r} + \frac{Q^2 + P^2}{r^2}$$

Properties: (neglect P).

1) $r \to \infty$, asymptotic flat

2) static, spherically symmetric, uniqueness

3) $r=0$ R_4 divergent, intrinsic singularity.

4) horizon ∞ - redshift

$\Delta = 0.$

$r^2 - 2GMr + Q^2 = 0.$

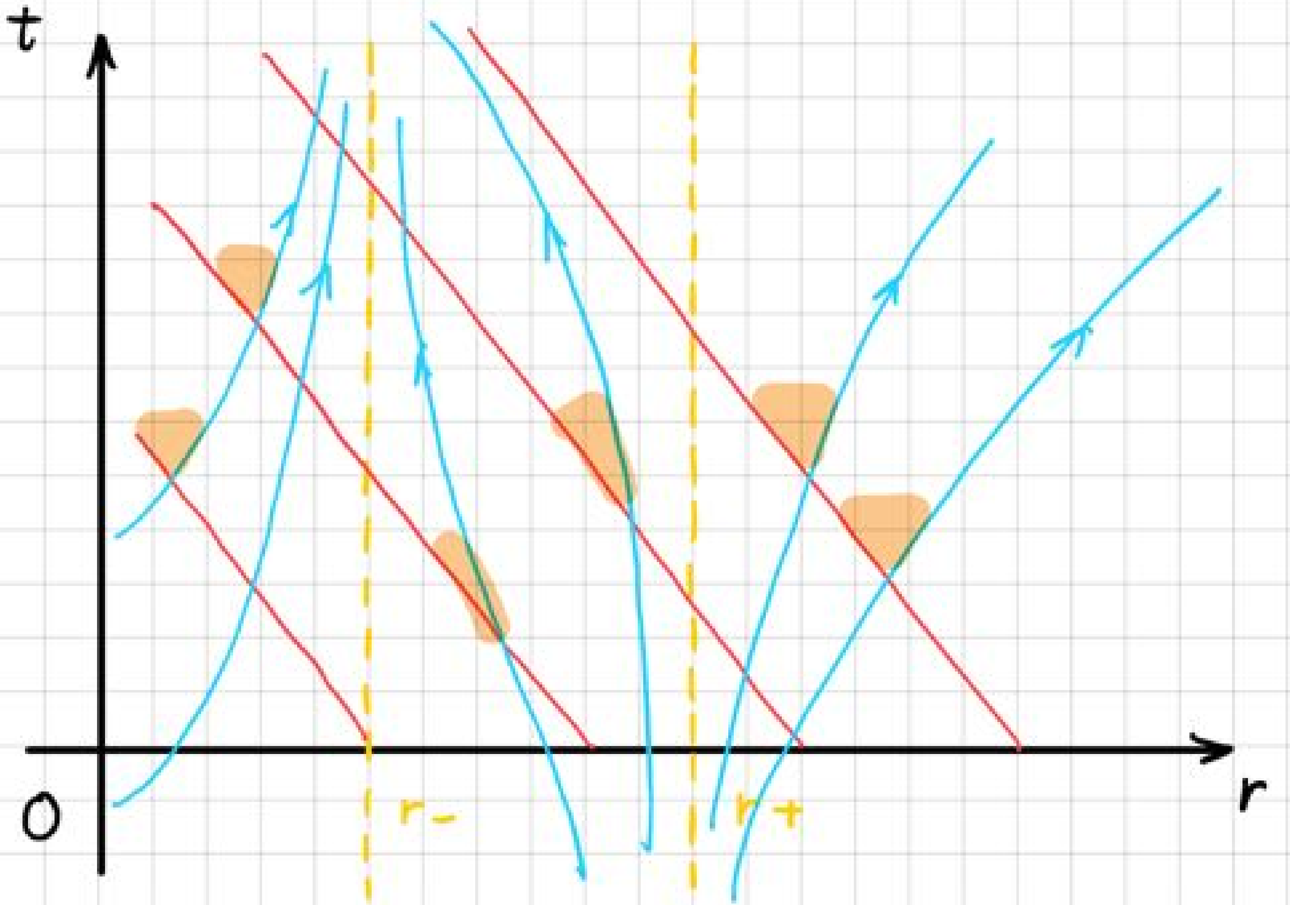
$r_\pm = GM \pm \sqrt{(GM)^2 - Q^2}$

i) $GM < Q$. no horizon,

naked singularity $\xrightarrow{\text{violate}}$ Cosmic censorship principle.

Non-physical

ii) $GM > Q$ $r = r_\pm$ $\begin{cases} \text{outer horizon } r = r_+ \\ \text{inner horizon } r = r_- \end{cases}$



∂t : Killing vector. $\partial\phi$: Killing vector.

$r > r_+$ t: time-like r: space-like

$r_- < r < r_+$ t: space-like r: time-like

$r < r_-$ t: time-like r: space-like

radially motion (uncharged). ∂t - Killing vector

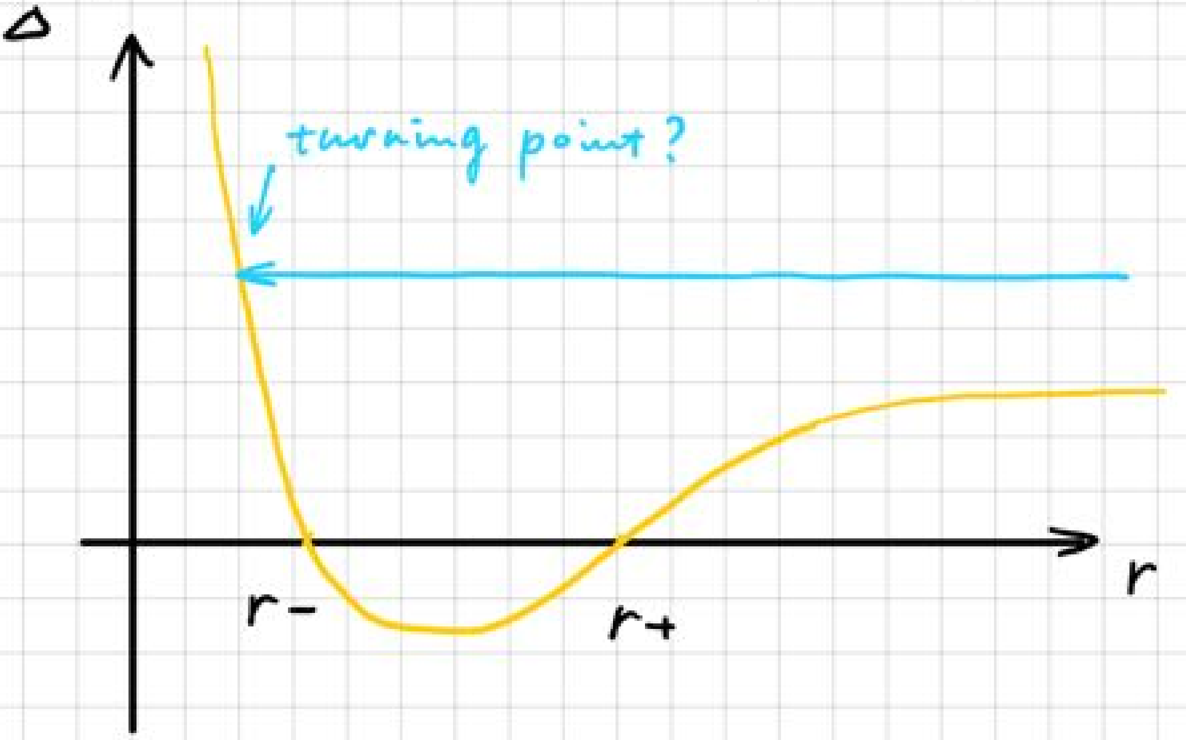
$E = -\hat{\xi}\cdot\hat{u}$

$= -\xi^0 u_0 \Rightarrow u_0 = -E$

kinetic potential

$\hat{u}\cdot\hat{u} = -1 \Rightarrow \Delta^{-1}E^2 + \left(\frac{dr}{d\tau}\right)^2\Delta^{-1} = -1 \Rightarrow \left(\frac{dr}{d\tau}\right)^2 + \Delta = u_0^2$

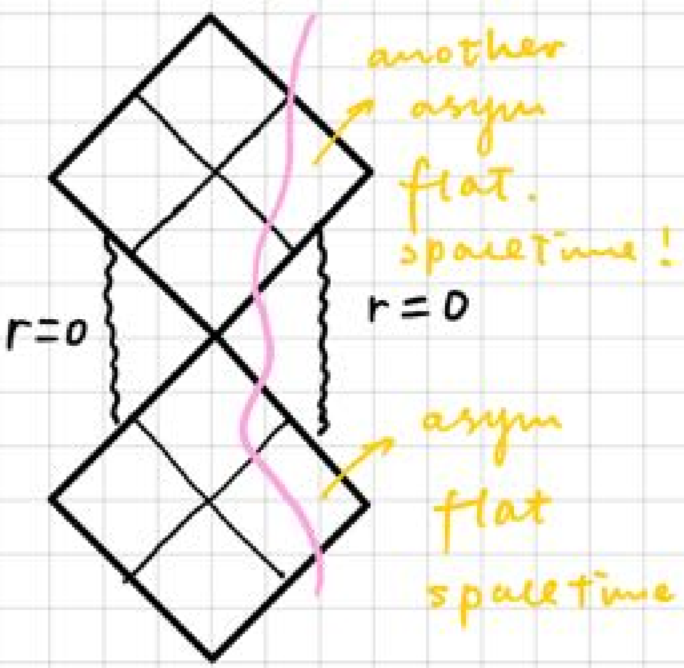
potential $\Delta = 1 - \frac{2GM}{r} + \frac{Q^2}{r^2}$



I) Seems there's always a turning point at $r < r_-$. However, you are not in the same asymptotic flat spacetime you enter!

Penrose diagram

$r=0$ time-like singularity.



3) $GM = Q$. extreme BH.

$r_+ = r_-$. when $GM \to Q$

$$ds^2 = ds^2_{AdS_2} + ds^2_{S^2}$$

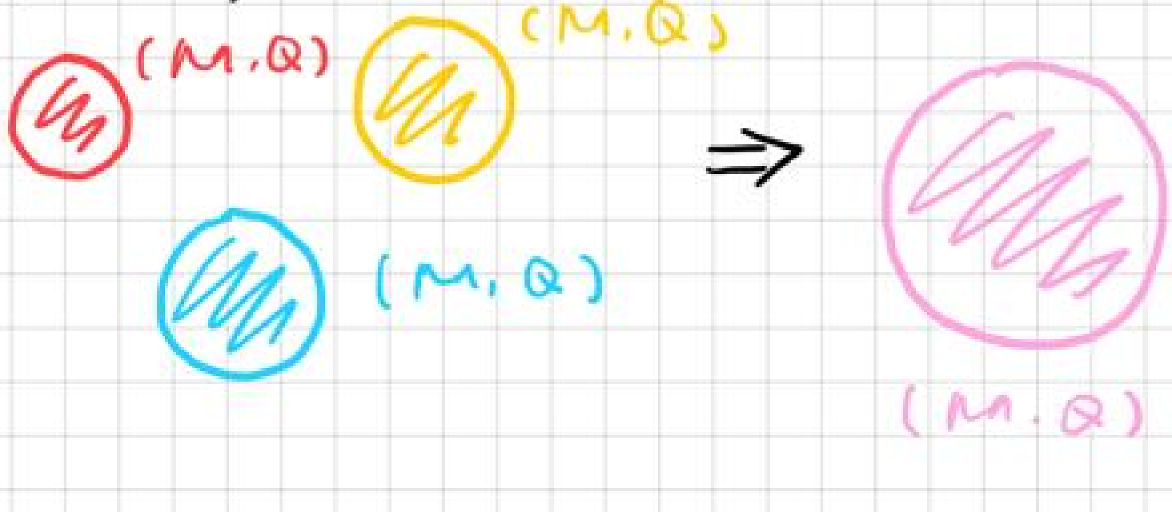
$$\begin{cases} AdS_2: \quad ds^2 = \dfrac{\ell^2}{r^2}\left(-dt^2 + dr^2\right) \\ S_2 \end{cases}$$

maximally symmetric spacetime.

Multi-center solutions.

$$ds^2 = ds^2(ERN_1) + ds^2(ERN_2) + \cdots$$

we find some solutions which can linearly combine. (physically, balance between gravity & EM)



Spacetime & rotation.

$\Omega \to$ spacetime outside.

1) free-dragging

2) gravito-mageuetic effect.

example, slowly notation.

Linear gravity.

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}.$$

Trace-reversed fluctuation. $\bar{h}_{\mu\nu} = 0$

$$\begin{cases} \Box \bar{h}_{\mu\nu} = -16\pi G T_{\mu\nu} \\ \partial^\mu \bar{h}_{\mu\nu} = 0 \end{cases}$$

$$T_{\mu\nu} = (\rho + p) u_\mu u_\nu + p g_{\mu\nu}$$

Dust $u^\mu = \gamma(1, v^i/c)$, $p \simeq 0$.

$T_{00} \sim \rho \quad T_{0i} \sim -\rho v/c \quad T_{ij} \sim \rho\left(\frac{v}{c}\right)^2$

$\Box \bar{h}_{00} = 16\pi G\rho \Rightarrow h_{00} = 2\Phi \quad h_{ij} = h_{00}\delta_{ij}$

$\Box \bar{h}_{0i} = -16\pi G T_{0i} \Rightarrow h_{0i} = \bar{h}_{0i} \equiv A_i$

Compare to EM

$\partial_t \rho = 0. \quad \partial_t h_{\alpha\beta} = 0.$

Geodesic Eq.

$$\Phi \sim \frac{GM}{r} \qquad v \sim \sqrt{\frac{GM}{r}} \qquad h_{ij} v \sim \left(\frac{GM}{r}\right)^{3/2}$$

$$\frac{dv_i}{dt} = -\left(\partial_i \Phi + \partial_t A_i\right) + \left(\partial_i A_l - \partial_l A_i\right) v_l$$

$$\begin{cases} \vec{E}_g = -\nabla\Phi - \partial_t \vec{A} & \text{gravitoelectric field} \\ \vec{B}_g = \nabla \times \vec{A} & \text{gravito magnetic field.} \end{cases}$$

$$\Downarrow$$

$$\begin{cases} \nabla \cdot \vec{B}_g = 0 \\ \nabla \times \vec{E}_g + \partial_t \vec{B}_g = 0. \end{cases}$$

Einstein Eq

minus $\Rightarrow$ attraction.

$$\Rightarrow \begin{cases} \nabla \cdot \vec{E}_g = -4\pi G\rho. \\ \nabla \times \vec{B}_g = -\dfrac{16\pi G}{c^2} \rho \vec{v} \end{cases}$$

another factor of 4

graviton ~ spin 2.

consider rotation. retarded time.

$$\bar{h}_{\mu\nu} = \frac{4G}{c^4}\int \frac{T_{\mu\nu}(t-|\vec{x}-\vec{y}|, y)}{|\vec{x}-\vec{y}|} d^3y.$$

$\downarrow$ static

$$= \frac{4G}{c^4}\int \frac{T_{\mu\nu}(y)}{|\vec{x}-\vec{y}|} d^3y.$$

$$\frac{1}{|\vec{x}-\vec{y}|} \approx \frac{1}{r} + \frac{x^i y^i}{r^3} + \cdots$$

$$\Phi(x) = -G\int \frac{\rho(y)}{|\vec{x}-\vec{y}|} d^3y \sim \frac{GM}{r}$$

$$A_i(x) = -\frac{4G}{c^2}\int \frac{T_{0i}}{|\vec{x}-\vec{y}|} d^3y$$

$$\sim \frac{1}{r}\left[\underbrace{\int T_{0i}\, d^3y}_{=0} + \frac{x^j}{r^2}\int y^j T_{0i}\, d^3y + \cdots\right]$$

energy-momentum conservation.

define:

$$J^k \equiv \int \varepsilon^{kij}(y^i T^{0j} - y^j T^{0i}) d^3y$$

$$= \frac{2G}{r^3}\varepsilon_{ijk}\, x^j J^k.$$

Metric

$$ds^2 = -(1+2\Phi)dt^2 + 2A_i\,dt\,dx^i + (1+2\Phi)\delta_{ij}\,dx^i dx^j$$

$$\parallel$$

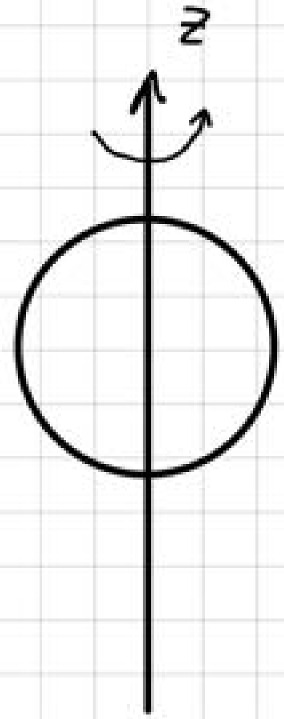
$$-\frac{4GJ}{c^3r^2}\sin^2\theta\,(r\,d\phi)(c\,dt)$$

$$J \sim I\Omega \sim MR^2\Omega$$

$$A \sim O(\Omega).$$

$$\frac{GJ}{c^3R^2} \sim \left(\frac{GM}{Rc^2}\right)\left(\frac{v}{c}\right)$$ gravito-magnetic effect

Lense-Thirring effect.



$$-\frac{4GJ}{c^3r^2}(c\,dt)\left(\frac{x\,dy - y\,dx}{r}\right)$$

a free-falling gyro

$$\begin{cases} u^\mu = (u^t\ 0\ 0\ u^z) \\ s^\mu = (0\ s^x\ s^y\ 0) \end{cases}$$

$$\hat{u}\cdot\hat{s} = 0.$$

no contribution from Φ. let $\Phi = 0$.

$$\Gamma^{x}{}_{ty} = \frac{2GJ}{c^2 z^3} \qquad (r = z)$$

$$\Gamma^{y}{}_{tx} = -\frac{2GJ}{c^2 z^3}$$

$$\begin{cases} \dfrac{dS^x}{dt} = -\dfrac{2GJ}{c^2 z^3} S^y \\ \dfrac{dS^y}{dt} = \underline{\dfrac{2GJ}{c^2 z^3}} S^x \end{cases}$$

$\Rightarrow$ precession, $\Omega = \dfrac{2GJ}{c^2 z^3}$

can add to de-Sitter Fokker

Kerr spacetime

what we want

- stationary ∂t
- axial symmetry. $\partial\phi$
- $g_{t\phi}$

$t \to -t$, $\phi \to -\phi$, rotation inv.

(t, x^i, x^j, ϕ)

$g_{01}=0 \quad g_{02}=0 \quad g_{13}=0 \quad g_{23}=0 \quad g_{03}\neq 0.$

$$ds^2 = g_{00}\,dt^2 + g_{33}\,d\phi^2 + 2g_{03}\,dt\,d\phi$$

$$+ \underset{\downarrow}{\gamma_{ij}}\,dx^i dx^j$$

$$e^{\Omega(x)}\,dx^i dx^j$$

$$\Rightarrow ds^2 = -A\,dt^2 + B(d\phi - \omega\,dt)^2 + (r,\theta)$$

$$= (-A + B\omega^2)\,dt^2 + B\,d\phi^2 - 2B\omega\,d\phi\,dt + \cdots$$

$$\Rightarrow \frac{g_{t\phi}}{g_{\phi\phi}} = -\omega.$$

Stationary limiting surface.

photon. fixed (r, θ).

$$ds^2 = 0 = g_{tt}\,dt^2 + 2g_{t\phi}\,dt\,d\phi + g_{\phi\phi}\,d\phi^2 = 0.$$

$$\left(\frac{d\phi}{dt}\right)^2 g_{\phi\phi} + 2\left(\frac{d\phi}{dt}\right) g_{t\phi} + g_{tt} = 0.$$

$$\left(\frac{d\phi}{dt}\right) = -\frac{g_{t\phi}}{g_{\phi\phi}} \pm \sqrt{\left(\frac{g_{t\phi}}{g_{\phi\phi}}\right)^2 - \frac{g_{tt}}{g_{\phi\phi}}}$$

$g_{\phi\phi} > 0$. if $g_{tt} < 0$, $\left(\frac{d\phi}{dt}\right)$ 2 solution > 0 & < 0.

if $g_{tt} > 0$. 2 solution in same direction.

$$g_{tt} = 0. \quad \frac{d\phi}{dt} = \begin{cases} -2\frac{g_{t\phi}}{g_{\phi\phi}} \\ 0 \end{cases}$$

SLS.

$\Rightarrow g_{tt} > 0$ no particle can be stationary.

$(u^\mu = (u^t, 0, 0, 0), \ \hat{u}\cdot\hat{u} > 0 \neq -1)$

also a ∞-redshift surface. $\frac{\nu_R}{\nu_E} \sim \left(\frac{g_{tt}}{g_{\phi\phi}}\right)^{1/2}$.

Event Horizon ? $\neq$ SLS

$\downarrow$

null surface. $\Rightarrow g^{rr}=0$.

in Schwarzchild spacetime $g_{tt}=g^{rr}=0$.

Kerr 1963.

Carter-Robinson theorem.

(M, g). asymptotic flat. static.

axial symmetry. at horizon, non-singular.

$\Rightarrow$ twoparameter Kerr spacetime (M, J).

Hawking-Wald BH

stationary $\Rightarrow$ axially symmetric.

+EM $\rightarrow$ (M, J, Q)

Kerr-Neumann spacetime.

Kerr spacetime. Boyer-Linquist coordinate

$$ds^2 = -dt^2 + \frac{\rho^2}{\Delta} dr^2 + \rho^2 d\theta^2 + (r^2 + a^2)\sin^2\theta \, d\phi^2$$

$$+ \frac{2GMr}{\rho^2}\left(a\sin^2\theta \, d\phi - dt\right)^2$$

$R_{\mu\nu} = 0$. solution to vaccum Zinstein eq^s.

$$\begin{cases} \Delta(r) = r^2 - 2GMr + a^2 \\ \rho(r) = r^2 + a^2\cos^2\theta . \end{cases}$$

$a = J/M$. Kerr parameter

$$g_{tt} = -1 + \frac{2GMr}{\rho^2}$$

$$= -\frac{r^2 + 2GMr + a^2\cos^2\theta}{\rho^2}$$

$$g_{rr} = \frac{\Delta}{\rho^2}$$

event horizon $\Rightarrow \Delta = 0$.

$r \to \infty$. Asymptotic flat.

$a \ll 1$. $\rho^2 = r^2$ $\Delta = r^2 - 2GMr$.

$$ds^2 = -dt^2 + \frac{r^2}{r^2 - 2GMr} dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2$$

$$+ \frac{2GM}{r}\left(-2a\sin^2\theta dt d\phi + dt^2\right)$$

$$= -\left(1 - \frac{2GM}{r}\right)dt^2 + \left(1 - \frac{2GM}{r}\right)^{-1} dr^2$$

$$+ r^2 d\Omega^2 - 4GJ$$

$M \to 0$ $d^2 S_{kerr} \to d^2 S_{Mink}$?

in B-L coordinate

$$ds^2 = -dt^2 + \frac{\rho^2}{r^2 + a^2} dr^2 + \rho^2 d\theta^2 + (r^2 + a^2)\sin^2\theta d\phi^2$$

use Kerr-Schild coordinate

$$\begin{cases} x = \sqrt{r^2 + a^2} \sin\theta \cos\phi \\ y = \sqrt{r^2 + a^2} \sin\theta \sin\phi \\ z = r \sin\theta . \end{cases}$$

rewrite the metric as

$$ds^2 = -\frac{\rho^2\Delta}{\Sigma^2}dt^2 + \frac{\Sigma^2\sin^2\theta}{\rho^2}(d\phi - \omega dt)^2$$

$$+\frac{\rho^2}{\Delta}dr^2 + \rho^2 d\theta^2$$

where

$$\Sigma^2 = (r^2+a^2)^2 - a^2\Delta\sin^2\theta$$

$$\omega = 2\mu r a/\Sigma$$

Observors.

1. ZAMO (zero angular momentum observor).

$$L = 0$$

$$\parallel$$

$$\hat{u}_0 \cdot \hat{\eta} = g_{t\phi} u^t_0 + g_{\phi\phi} u^\phi_0$$

$$\Rightarrow \Omega_0 = \left.\frac{d\phi}{dt}\right|_0 = \frac{u^\phi_0}{u^t_0} = -\frac{g_{t\phi}}{g_{\phi\phi}}$$

a ZAMO actually show an angular velocity!

$\Rightarrow$ Free-dragging effect $\omega_H = \frac{2\mu r + a}{(r_+^2 + a^2)^2} = \frac{a}{2\mu r_+}$

2. Static observor.

$$u^\mu = \gamma (1, 0, 0, 0).$$

$$\hat{u} \cdot \hat{u} = -1$$

$$\gamma = \sqrt{-g_{tt}}.$$

well defined when $g_{tt} > 0 \Rightarrow r > r_{S+}$

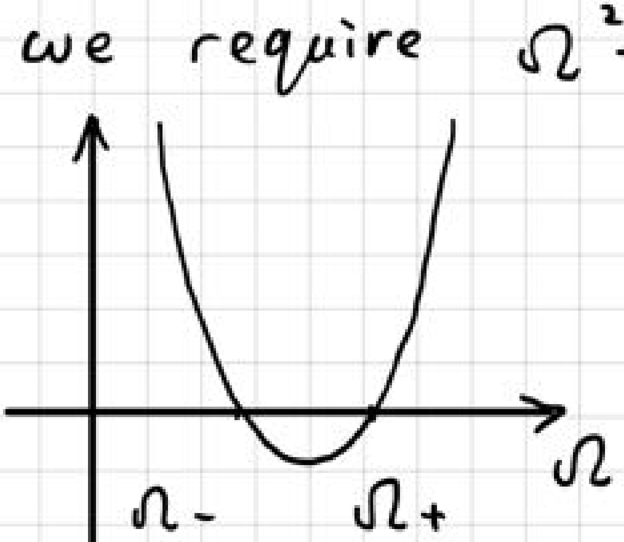
no static observor at $r \leq r_{S+}$

3. Stationary observor (r, θ) fixed.

$$u^\mu = \gamma (\xi^\mu + \Omega \eta^\mu) = \gamma (1 \; 0 \; 0 \; \Omega)$$

$$\hat{u} \cdot \hat{u} = -1 \Rightarrow \gamma^{-2} = -g_{\phi\phi} \left(\Omega^2 - 2\omega\Omega + \frac{g_{tt}}{g_{\phi\phi}} \right)$$

we require $\Omega^2 - 2\omega\Omega + \frac{g_{tt}}{g_{\phi\phi}} < 0$



$$\Omega_\pm = \omega \pm \frac{\Delta^{1/2} \rho^2}{\Sigma^2 \sin\theta}.$$

$$\Omega_\pm = \omega \pm \frac{\Delta^{1/2}\rho^2}{\Sigma^2 \sin\theta}.$$

$\Omega = 0$ static $r > r_{s+}$

from $r_{s+} \to r_+$ $\Omega_- \uparrow$ $\Omega_+ \downarrow$

finally Ω_- meets Ω_+ at r_+

when $\Omega_- = \Omega_+ = \omega_H$.

Motion in Kerr spacetime.

here we only consider $\theta = \frac{\pi}{2}$. $(u^\theta = 0)$.

(otherwise it's too complex).

$$\begin{cases} \mathcal{E} = -\hat{\xi} \cdot \hat{u} \\ L = \hat{\eta} \cdot \hat{u} \end{cases}$$

$$\Rightarrow \begin{cases} -\mathcal{E} = g_{tt} u^t + g_{\phi\phi} u^\phi \\ L = g_{t\phi} u^t + g_{\phi\phi} u^\phi \end{cases}$$

$$\Rightarrow \begin{cases} u^t = \frac{1}{\Delta}\left((r^2+a^2+2GMa^2/r)\mathcal{E} - \frac{2GMa}{r}L\right) \\ u^\phi = \frac{1}{\Delta}\left((1-2GM/r)L + \frac{2GMa}{r}\mathcal{E}\right). \end{cases}$$

from $\hat{u}\cdot\hat{u} = -1$ we finally show

$$\frac{\mathcal{E}^2-1}{2} = \frac{1}{2}\left(\frac{dr}{d\tau}\right)^2 + V_{eff}(r,\mathcal{E},L).$$

affected by rotational direction!

$$V_{eff} = -\frac{GM}{r} + \frac{L^2 - a^2(\mathcal{E}^2-1)}{2r^2} - \frac{GM(L-a\mathcal{E})^2}{r^3}.$$

Circular orbit

$$\frac{\partial V}{\partial r} = 0 \quad \frac{\partial^2 V}{\partial r} < 0.$$

ISCO. $r=\mu$. co-rotating.

$r = 6\mu$. Sch BH.

$r = 9\mu$ counter-rotating.

for a particle released at ∞.

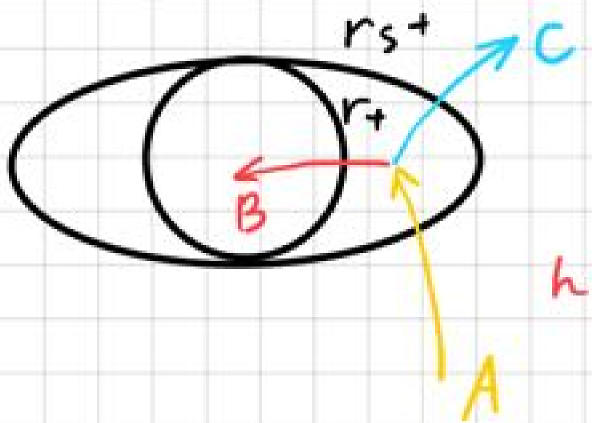
migrate to ISCO. energy loss ~ 44%.

Null geodesic

$$\frac{1}{L^2}\left(\frac{dr}{d\lambda}\right)^2 = \frac{1}{b^2} - V_{eff}(r, b, \sigma)$$

$$V_{eff} = \frac{1}{r^2}\left(1 - \left(\frac{a}{b}\right)^2 - \frac{2\mu}{r}\left(1 - \sigma\frac{a}{b}\right)^2\right)$$

Penrose process. $\mathcal{E}_C > \mathcal{E}_A$?



* it ensures.

r_+ always keeps

happens at D

A: $\mathcal{E}^{(A)} = -\hat{p}^{(A)}(\Sigma)\cdot\hat{\xi}$

$= -p^A{}_t(\Sigma)$

can go to ∞

$p^A{}_t(\Sigma) = p^A{}_t(D) \quad p^C_t(\Sigma) = p^C{}_t(D).$

$\mathcal{E}^{(C)} = p^B_t(D) + \mathcal{E}^{(A)}$

$p^B{}_t(D) = \hat{p}^{(B)}\cdot\hat{\xi}$

if

i) B finally get out of r_{s+}.

$$\xi^C = -\xi^B + \xi^A . \Rightarrow \xi^C < \xi^A .$$

ii) B stays $r < r_{s+}$. $g_{tt} > 0$.

$\hat{\xi}$ spacelike. $p_t^{(B)} > 0$ is possible.

$$\Rightarrow \xi^C > \xi^A .$$

from an observer at ∞. the particle is energized $\rightarrow$ BH $M \downarrow$

$$\begin{cases} M \rightarrow M - p_t^B \\ J \rightarrow J + p^B_\phi \end{cases}$$

we want to show $p^B_\phi < 0$. otherwise it may violate cosmic censorship principle.

consider an observor inside ergosphere.

at fixed (r, θ). $\hat{u}_0 = u^t(\hat{\xi} + \Omega\hat{\eta})$.

$$z^{(A)}\big|_0 = -\hat{p}^B \cdot \hat{u}_0 = -u^t\left(p^B{}_t + \Omega p^B{}_\phi\right) > 0.$$

$$\Rightarrow p^B{}_t + \Omega p^B{}_\phi < 0$$

$$\Rightarrow p^B{}_\phi < -\frac{p^B{}_t}{\Omega}$$

$\Rightarrow \delta M < 0$, $\delta J < 0$, and $\delta M > \Omega \delta J$.

However for $A = 8\pi\mu\left(\mu + \sqrt{\mu^2 - a^2}\right)$

$$\delta A = \frac{8\pi}{\kappa}\left(\delta M - \Omega_M \delta J\right)$$

$$\kappa = \frac{\sqrt{M^2 - a^2}}{2Mr_+} \qquad \Omega_M = \frac{a}{2\mu r_+}$$

$\uparrow$

surface gravity

Introduce irreducible mass

$$M_{ir} = \left(\frac{A}{16\pi}\right)^{1/2}, \quad M^2 = M_{ir}^2 + \frac{J^2}{4M_{ir}^2}$$

compare with 1^{st} law of Thermodynamics

$$dU = T dS + \cdots$$

$$\Rightarrow dS = \frac{1}{T} dU + \cdots$$

$$\begin{cases} A \sim S \\ T \sim K \end{cases}$$

	BH	Thermodynamics.
0^{th}	$K = \text{const}$	$T = \text{const}$
1^{st}	$\delta M = \frac{8\pi}{K}(\delta A + \Omega_H \delta J)$	$dU = T dS + \cdots$
2^{nd}	$\delta A > 0$	$\delta S > 0.$
3^{rd}	extreme BH ($K = 0$) does not exist	cannot reach $T = 0K$ with finite procedures.

The entropy of BH ?

$$\delta(S_{BH} + S_{ex}) > 0.$$

Bekenstein $\quad S_{BH} \propto A.$

a quantum thing!

Hawking $\quad S_{BH} = \frac{k_B A}{4 \hbar G}$

Holography principle.

The internal physics can be discussed with physics at horizon

Hawking Temperature.

$$T_H = \frac{\hbar K}{2\pi k_B}$$

$\Rightarrow$ blackbody radiation?

$\exists$ Hawking radiation

Stefan-Boltzmann. law $\sigma T^4 \propto \frac{1}{M^4}$

$\downarrow$

small BHs are much easier to vanish.

Gravitational Wave

Frame work of weak field approximation

propagation degrees of freedom

$16 - 10 - 4 = 2.$ $\qquad \frac{D(D-3)}{2}$ no GW in 3D.

$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}.$

use trace-reverse perturbation.

$$\bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu} h.$$

$\Rightarrow \partial^\mu \bar{h}_{\mu\nu} = 0 \qquad (\sim \partial^\mu A_\mu = 0)$

if we choose $h_{\mu\nu} \to \bar{h}'_{\mu\nu} = \bar{h}_{\mu\nu} - \partial_\mu \xi_\nu - \partial_\nu \xi_\mu$

$$+ \eta_{\mu\nu}\partial_\lambda \xi^\lambda$$

$\partial^\mu \bar{h}'_{\mu\nu} = \partial^\mu \bar{h}_{\mu\nu} - \Box \xi_\nu$ not unique!

we can choose ξ_ν to satisfy $\partial^\mu \bar{h}'_{\mu\nu} = 0.$

$\longrightarrow$ remaining Dof. if $\exists\, \zeta_\nu,\ \Box \zeta_\nu = 0$ $\vec{\xi}' \to \vec{\xi} + \vec{\zeta}$
s.t $\partial^\mu \bar{h}'_{\mu\nu} = 0$

GW in vaccum.

$\Box \bar{h}_{\mu\nu} = 0.$ $\Box = \eta_{\mu\nu} \partial^\mu \partial^\nu$

let $\bar{h}_{\mu\nu} = C_{\mu\nu} e^{ik^\sigma x_\sigma}$ → propagating at speed of light.

$\longrightarrow k^\mu k_\mu = 0$ null wave vector.

harmonic gauge $\partial^\mu \bar{h}_{\mu\nu} = 0$

$\Rightarrow k^\mu C_{\mu\nu} = 0$ → transverse wave.

→ 4 constraints.

let $x^{\mu'} \to x^\mu + \xi^\mu$. where ξ^μ satisfy

$\Box \xi^\mu = 0 \quad (\xi^\mu = B^\mu e^{i\hat{k}\cdot\hat{x}})$

$\Rightarrow C_{\mu\nu} \to C'_{\mu\nu} = C_{\mu\nu} - ik_\mu B_\nu - ik_\nu B_\mu + i\eta_{\mu\nu} k_\lambda B^\lambda$

Claim we can choose B_ν to have

$$\begin{cases} C'^\mu{}_\mu = 0 \ (1) & \text{traceless.} \\ C'_{0\mu} = 0 \ (2). \end{cases}$$

$(1) \Rightarrow C^{\mu}{}_{\mu} + 2i k_\lambda B^\lambda = 0$

$(2) \Rightarrow \begin{cases} B_0 = \cdots \\ B_j = \cdots \end{cases}$

Thus we have

$$\begin{cases} k^\mu C'_{\mu\nu} = 0 & 4 \\ C'^{\mu}{}_{\mu} = 0 & 1 \\ C'_{0\mu} = 0 & 4 \end{cases} \quad -1$$

in total 8 independent constraint.

Assume. $k^\mu = (\omega, 0, 0, k)$

$$\Rightarrow C_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & C_{11} & C_{12} & 0 \\ 0 & C_{12} & -C_{11} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \rightarrow \text{two polarizations}$$

Gravitational radiation

Transverse - Traceless.

TT-gauge. $\bar{h} = 0$. $(h_{\mu\nu} = \bar{h}_{\mu\nu})$.

when we have $h_{\mu\nu}$ in any other gauges.

can we project it to TT-gauge?

① $P_{\mu\nu} = \eta_{\mu\nu} + n_\mu n_\nu$

$\hat{n}$ wave direction vector.

$\rightarrow$ convert to transverse

② substract trace.

$\rightarrow$ TT

$$h^{TT}_{\mu\nu} = P^\sigma{}_\mu P^\rho{}_\nu h_{\sigma\rho} - \frac{1}{2} P_{\mu\nu} P^{\rho\sigma} h_{\rho\sigma}$$

Geodesic deviation eq.

$$\frac{d^2 S^\mu}{d\tau^2} = R^\mu{}_{\nu\rho\sigma} u^\nu u^\rho S^\sigma$$

$u^\nu = (1\ 0\ 0\ 0)$

$$\Rightarrow \frac{d^2 S^\mu}{d\tau^2} = R^\mu{}_{00\sigma} S^\sigma$$

$$= \frac{1}{2} S^\sigma \frac{\partial^2}{\partial \tau^2} h^\mu{}_\sigma$$

weak field $d\tau \approx dt$

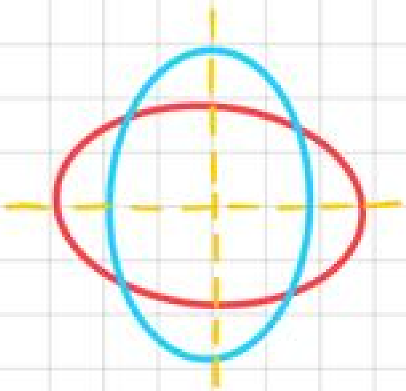
def $C_{11} \equiv C_+ \quad C_{12} \equiv C_\times$

let $C_+ \neq 0 \quad C_\times = 0.$

$$\begin{cases} \frac{\partial^2}{\partial t^2} s^1 = \frac{1}{2} s^1 \frac{\partial^2}{\partial t^2} \left(C_+ e^{ikx} \right) \\ \frac{\partial^2}{\partial t^2} s^2 = -\frac{1}{2} s^2 \frac{\partial^2}{\partial t^2} \left(C_+ e^{ikx} \right). \end{cases}$$

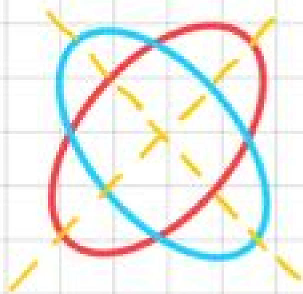
$$\Rightarrow s^1 = \left(1 + \frac{1}{2} C_+ e^{ikx} \right) s^1(0)$$

$$s^2 = \left(1 - \frac{1}{2} C_+ e^{ikx} \right) s^2(0)$$



let $C_+ = 0, \quad C_\times \neq 0.$

$$\begin{cases} \frac{\partial^2}{\partial t^2} s^1 = \frac{1}{2} s^2 \frac{\partial^2}{\partial t^2} \left(C_\times e^{ikx} \right) \\ \frac{\partial^2}{\partial t^2} s^2 = \frac{1}{2} s^1 \frac{\partial^2}{\partial t^2} \left(C_\times e^{ikx} \right) \end{cases}$$



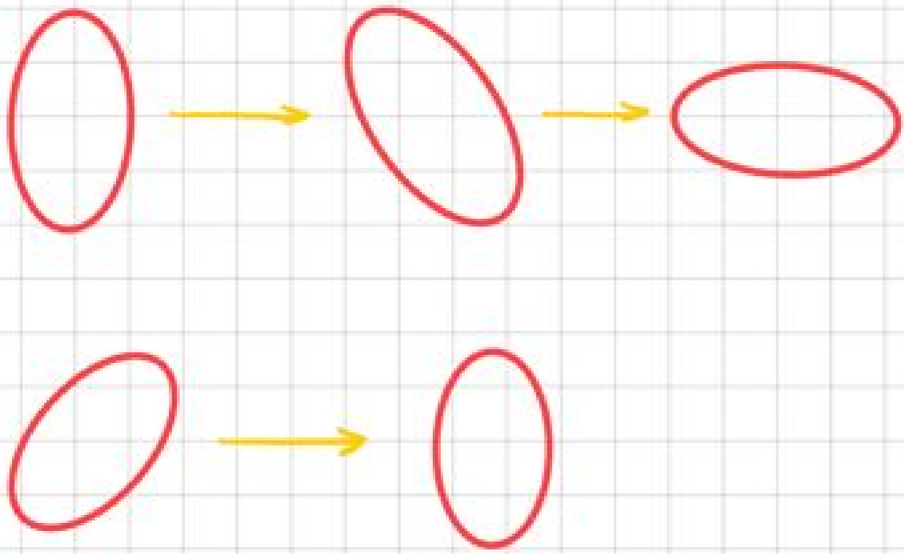
$$\Rightarrow \begin{array}{l} s^1 = s^1(0) + \frac{1}{2} C_\times e^{ikx} s^2(0) \\ s^2 = s^2(0) + \frac{1}{2} C_\times e^{ikx} s^1(0). \end{array}$$

circular polarization

$$C_R = \frac{1}{\sqrt{2}}(C_+ + iC_\times)$$

$$C_L = \frac{1}{\sqrt{2}}(C_+ - iC_\times).$$

C_R:



rotate $180^\circ \longrightarrow$ back to itself.

$\Downarrow$

spin 2

(like fermion, spin $\frac{1}{2}$, rotate 360°,

$\psi \mapsto -\psi$, rotate 720°, $\psi \mapsto \psi$).

$SO(1, D-1)$ Lorentzian group.

Massless $SO(1, D-2)$.

symmetric & traceless representation

$$\frac{(D-2)(D-1)}{2} - 1 = \frac{D(D-3)}{2}$$

GW Source

$$\Box \bar{h}_{\mu\nu} = -16\pi G T_{\mu\nu}$$

$\sim \Box A_\mu \propto J_\mu.$

- linear eq
- Green's function (retarded).

$$\bar{h}_{\mu\nu}(x^0) = -16\pi G \int G_{rad}(x^0 y^0) T_{\mu\nu}(y^0) dy^0$$

$$= \frac{G}{4} \int \frac{1}{|\vec{x}-\vec{y}|} T_{\mu\nu}\big(\underbrace{t-|\vec{x}-\vec{y}|}_{\text{retarded time } t_r}\big) d^3y.$$

Far Zone Source.

- Isolated.
- Far away
- Non-relativistic.

Fourier transform.

$$\tilde{\phi}(\omega, \vec{x}) = \frac{1}{\sqrt{2}\pi} \int dt\, e^{i\omega t} \phi(t, \vec{x})$$

$$\Rightarrow \tilde{\bar{h}}_{\mu\nu}(\omega, \vec{x}) = \frac{4G}{\sqrt{2}\pi}\int dt\, d^3y \frac{T_{\mu\nu}(t-|\vec{x}-\vec{y}|)}{|\vec{x}-\vec{y}|.} e^{i\omega t}.$$

$dt \to dt_r$

$$= \frac{4G}{\sqrt{2}\pi}\int dt_r\, d^3y \frac{T_{\mu\nu}(t_r)}{|\vec{x}-\vec{y}|} e^{i\omega(t_r+|\vec{x}-\vec{y}|)}$$

$$= \frac{4G}{\sqrt{2}\pi}\int d^3y\, e^{i\omega|\vec{x}-\vec{y}|} \frac{\tilde{T}_{\mu\nu}(\omega, \vec{y})}{|\vec{x}-\vec{y}|}$$

far zone approximation. $|\vec{x}-\vec{y}| \simeq R$.

slowly moving source (ω small enough, length scale of source >> GW wavelength).

$$\approx \frac{4G}{\sqrt{2}\pi}\frac{e^{i\omega R}}{R}\int d^3y\, \tilde{T}_{\mu\nu}(\omega, \vec{y})$$

harmonic gauge + $\partial^\mu T_{\mu\nu} = 0$.

$\partial^\mu \bar{h}_{\mu\nu} = 0$.

$\Downarrow$

$\tilde{\bar{h}}^{0\nu} = \frac{1}{\omega}\partial_i \tilde{\bar{h}}^{i\nu}$

only consider $\tilde{\bar{h}}^{ij}$!

$\tilde{\bar{h}}^{ij} \to \tilde{\bar{h}}^{i\nu} \to \tilde{\bar{h}}^{0\nu}(\tilde{\bar{h}}^{\mu\nu})$

$$\int d^3y\, \tilde{T}^{ij}(\omega, \vec{y})$$

$$= \int \underbrace{\partial_k (y^i T^{kj})}_{=0} d^3y - \int y^i \partial_k \tilde{T}^{kj}(\omega, \vec{y})\, d^3y$$

use $\partial_\mu T^{\mu\nu} = 0 \Rightarrow -\partial_k \tilde{T}^{k\mu} = i\omega \tilde{T}^{0\mu}$.

$$= i\omega \int y^i \tilde{T}^{0\mu}(\omega, \vec{y})\, d^3y.$$

$$= \frac{i\omega}{2} \int \left(y^i \tilde{T}^{0j} + y^j \tilde{T}^{0i} \right) d^3y$$

$$= \frac{i\omega}{2} \left[\int \underbrace{\partial_l (y^i y^j \tilde{T}^{0l})}_{=0} d^3y - \int y^i y^j \partial_l \tilde{T}^{0l} d^3y \right]$$

energy density.

$$= -\frac{\omega^2}{2} \int y^i y^j \tilde{T}^{00} d^3y.$$

Def: quadrapole moment tensor.

$$q_{ij}(t) = \int y^i y^j T^{00}(t, \vec{y})\, d^3y.$$

$$\Rightarrow \tilde{\bar{h}}(\omega, \vec{x}) = -2G\omega^2 \frac{e^{i\omega R}}{R} \tilde{q}_{ij}(\omega)$$

$$\bar{h}_{ij}(t, \vec{x}) = -\frac{1}{\sqrt{2\pi}} \frac{2G}{R} \int d\omega\, e^{-i\omega(t-R)} \omega^2 \tilde{q}_{ij}(\omega)$$

$$= \frac{1}{\sqrt{2\pi}} \frac{2G}{R} \frac{d^2}{dt^2} \int d\omega\, e^{-i\omega t_r} \tilde{q}_{ij}(\omega)$$

$$= \frac{2G}{R} \frac{d^2}{dt^2} q_{ij}(t_r)$$

Binary system. circular motion.

$$\frac{GM^2}{(2r)^2} = M\frac{v^2}{r} \Rightarrow v = \left(\frac{GM}{4r}\right)^{1/2}$$

$$\Rightarrow T = \frac{2\pi r}{v}$$

$$\Omega = \frac{2\pi}{T}$$

$x_a = r(\cos\Omega t, \sin\Omega t)$

$x_b = r(-\cos\Omega t, -\sin\Omega t)$.

$$T^{00}(t, \vec{x}) = M\left[\delta(x^3)\delta(x^1 - x_a^1)\delta(x^2 - x_a^2) + \delta(x^3)\delta(x^1 - x_b^1)\delta(x^2 - x_b^2)\right].$$

$$\begin{cases} q_{11} = Mr^2(1+\cos 2\Omega t) \\ q_{22} = Mr^2(1-\cos 2\Omega t) \\ q_{21} = -q_{21} = Mr^2 \sin 2\Omega t. \end{cases}$$

$$T_{00} \longrightarrow q^{ij} \longrightarrow \bar{h}^{ij} \longrightarrow \bar{h}^{0i} \longrightarrow \bar{h}^{\mu\nu}.$$

$$\begin{cases} \text{EM radiation} & \Box \bar{h}^{\mu\nu} = -16\pi G T^{\mu\nu}. \\ \text{GW radiation.} & \Box A^{\mu} = -4\pi J^{\mu} \end{cases}$$

$\longrightarrow$ Multipole expansion / Green function.

$$A^{\mu} = \int d^3y \, \frac{e^{ik|\vec{x}-\vec{y}|}}{|\vec{x}-\vec{y}|}$$

$$|\vec{x}-\vec{y}| = r - \hat{n}\cdot\vec{y}$$

$$\frac{e^{ikr}}{r} \sum_m \frac{(-ik)^m}{m!} \int d^3y \, J^{\mu}(y) (\hat{n}\cdot\vec{y})^m.$$

$$= \frac{1}{r} \sum_l \frac{d^l}{dt^l} (E_l + M_l).$$

where $Z_l \propto \int y^l \rho(y) d^3y$

$$M_l \propto \int y^l \rho(y) \left(\frac{v}{c}\right) d^3y .$$

in GR.

similarly.

$h^\mu{}_l \sim \frac{1}{r} \frac{d^l}{dt^l} M_l \quad M_l \sim \int y^l \rho(y) d^3y.$ Mass multipole

$h^s{}_l \sim \frac{1}{r} \frac{d^l}{dt^l} S_l \quad S_l \sim \int y^l v \rho(y) d^3y.$ current multipole

$l=0$, $M_0 \rightarrow$ total mass $\rightarrow$ conserved.

$S_0 \rightarrow$ total momentum (conserved)

$l=1$ $M_1 \rightarrow$ in mass center frame $=0$.

$S_1 \rightarrow$ total angular momentum (conserved)

$l=2$ $M_2 \rightarrow$ mass quadrupole (first term) since $\bar{h}^{\mu\nu}$ is dimension less

$$v \to \left(\frac{v}{c}\right) \qquad \text{Mass} \to \left(\frac{G}{c^2}\right)$$

$$h_l^M \sim \frac{1}{r}\frac{G}{c^{l+2}}\frac{d^l}{d^{l+1}} M_l$$

$$h_l^S \sim \frac{1}{r}\frac{G}{c^{l+3}}\frac{d^l}{d^{l+1}} S_l \leftarrow \text{depressed by } 1/c$$

$$\to \bar{h}^{ij} = \frac{2G}{Rc^4}\frac{d^2}{dt^2} q^{ij} \sim \frac{2G}{Rc^4} M v^2$$

if we choose $R \sim 1\,\text{Mpc}$. $Mv^2 \sim M_\odot c^2$

$$\Rightarrow \bar{h}^{ij} \sim 10^{-21} - 10^{-22}$$

How to define the energy - momentum tensor of GW?

$g_{\mu\nu}$ dynamic.

$\left\{\begin{array}{l}\phi \\ A_\mu\end{array}\right.$ are usually defined in a flat spacetime.

however here we only consider

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}.$$

$\downarrow$

symmeoric tensor field.

$$g_{\mu\nu} = \eta_{\mu\nu} + \underset{\substack{\wr \\ O(\epsilon)}}{h^{(1)}_{\mu\nu}} + \underset{\substack{\wr \\ O(\epsilon^2)}}{h^{(2)}_{\mu\nu}}$$

$\downarrow$

$$G_{\mu\nu} = 8\pi G T_{\mu\nu}$$

$$O(\epsilon) \Rightarrow G^{(1)}_{\mu\nu}(h^{(1)}{}_{\mu\nu}) = 0.$$

$$O(\epsilon^2) \Rightarrow G^{(1)}{}_{\mu\nu}(h^{(2)}{}_{\mu\nu}) = -G^{(2)}_{\mu\nu}(h^{(1)}_{\mu\nu})$$

$$\equiv 8\pi G t_{\mu\nu}$$

Pros: $\begin{cases} t_{\mu\nu} \text{ symmetric} \\ \partial^\mu t_{\mu\nu} = 0 \end{cases}$

Cons: not gauge inv.

Solution of not gauge inv.

local (if local one can choose RZC).

$\Downarrow$

average on a few wavelength of GW

$\Downarrow$

$\langle t_{\mu\nu} \rangle$. we assume $\langle \partial_\mu (X) \rangle = 0$.

$\downarrow$

$$\langle A \partial_\mu B \rangle = - \langle B \partial_\mu A \rangle$$

choose TT-gauge.

$$\langle R^{(2)}_{\mu\nu} \rangle = -\frac{1}{4} \langle (\partial_\mu h_{\rho\sigma})(\partial_\nu h^{\rho\sigma}) \rangle - \frac{1}{2} \eta^{\rho\lambda} (\Box h_{\rho\sigma}) h^{\rho\sigma}$$

TT-gauge $\Box h^{TT}{}_{\rho\sigma} = 0$.

$$G^{(2)}{}_{\mu\nu} = R^{(2)}{}_{\mu\nu} - \frac{1}{2} \eta_{\mu\nu} \underset{\overset{\shortparallel}{0}}{R^{(2)}}$$

$$\Rightarrow t_{\mu\nu} = \frac{1}{32\pi G} \langle \partial_\mu (h_{\rho\sigma}^{TT}) \partial_\nu (h^{\rho\sigma}_{TT}) \rangle$$

$$h_{\mu\nu}^{TT} = C_{\mu\nu} e^{i\hat{k}\cdot\hat{x}} \quad \text{or} \quad C_{\mu\nu} \sin(\hat{k}\cdot\hat{x}).$$

$$t_{\mu\nu} = -\frac{1}{32\pi G} k_\mu k_\nu C_{\rho\sigma} C^{\rho\sigma} \underbrace{\langle \cos^2(k_\lambda x^\lambda) \rangle}_{= \frac{1}{2}}$$

$$k_\lambda = (\omega \quad 0 \quad 0 \quad \omega)$$

$$C_{\rho\sigma} C^{\rho\sigma} = 2(h_+^2 + h_\times^2)$$

$$t_{\mu\nu} = \frac{\pi}{8G} f^2 (h_+^2 + h_\times^2) \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & & & 0 \\ 0 & & & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix}$$

$$T_{0z} \ (\text{flux}) \sim 10^{-6} \left(\frac{f}{\text{Hz}}\right)^2 \frac{h_+^2 + h_\times^2}{(10^{-22})^2} \frac{\text{erg}}{\text{cm}^2 \cdot \text{s}}.$$

$$q^{ij} \longrightarrow \bar{h}^{ij}$$

Traceless $\longrightarrow$ reduced quadrupole.

$$\underline{J^{ij}} = q^{ij} - \frac{1}{3}\delta^{ij} (\delta_{kl} q^{kl})$$

↳ traceless.

$$h_{ij}^{TT} = \frac{2G}{r} \frac{d^2 J^{TT}(t-r)}{dt^2}$$

Total energy loss $\Delta E = \int_{\text{at } \infty} T_{0r}$.

$$t_{0r} = -\frac{G}{8\pi r^2} \left\langle \frac{d^3 J_{ij}^{TT}}{dt^3} \cdot \frac{d^3 J^{ij}_{TT}}{dt^3} \right\rangle$$

$$\Delta E = -\int P\, dt$$

$$P = \int_{r \to \infty} t_{0\mu} n^{\mu} r^2 d\Omega.$$

$$= \int_{r \to \infty} t_{0r} r^2 d\Omega$$

so far we do not ensure that J^{ij} is

transverse $\rightarrow$ projection.

$$P_{ij} = \delta_{ij} - n_i n_j$$

$$X_{ij}^{TT} = \left(P^k{}_i P^l{}_j - \frac{1}{2} P_{ij} P^{kl} \right) X_{kl}$$

use $\int d\Omega = 4\pi \quad \int n_i n_j d\Omega = \frac{4\pi}{3} \delta_{ij}$

$$\int n_i n_j n_k n_l \, d\Omega = \frac{4\pi}{15} \left(n_i n_j \cdots \right)$$

$$P = -\frac{G}{5}\left\langle \frac{d^3 J_{ij}}{dt^3}\frac{d^3 J^{ij}}{dt^3}\right\rangle$$

circular binary.

$$J_{ij} = MR^3 \begin{pmatrix} \frac{1}{3}+\cos 2\Omega t & \sin 2\Omega t & \\ \sin 2\Omega t & 1-3\cos\Omega t & \\ & & -2 \end{pmatrix}$$

$$P = -\frac{128}{5} G M^2 R^4 \Omega^6 = -\frac{2}{5}\frac{G^4 M^5}{R^5}$$

$$\uparrow$$

$$\Omega = \left(\frac{GM}{4R^3}\right)^{1/2}$$

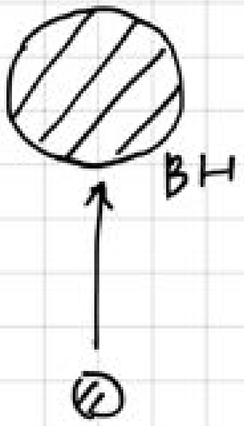
$$\Rightarrow \frac{dE}{dt} \Rightarrow \begin{cases} \frac{dR}{dt} \cdots \\ \frac{d\Omega}{dt} \cdots \end{cases}$$

Sources:

periodic rotating neutron star.

binary stars

burst compact star coalescence.



EMRI?

supernova

Observation.

10^0 Hz – 10^4 Hz LIGO & VIRGO stellar-mass Binary coalescence

10^{-4} Hz – 10^0 Hz SMBH coalescence.

EMRI LISA.

even lower

pular timing array. CMB.

final.

geodesic eq.

Christoffel.

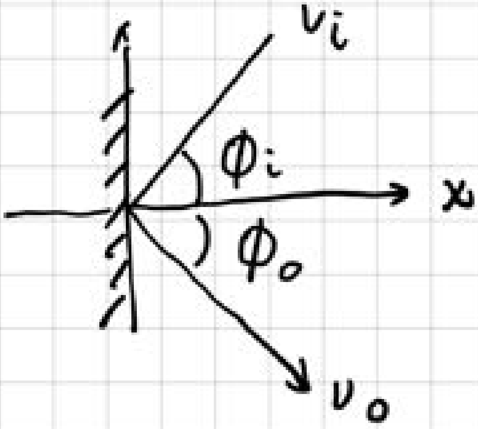
Exercise.

Chapter 1 . 13.

Chapter 2 . 2.

Chapter 3 . 5

Chapter 5 . 9 .



$$k^{\mu} = (\nu_i, -\nu_i\cos\phi, -\nu_i\sin\phi_i)$$

$$k'^{\mu} = (\nu_o, \nu_o\cos\phi_o, -\nu_o\sin\phi_o)$$

$$\Lambda = \begin{pmatrix} \gamma & -\beta\gamma & & \\ -\beta\gamma & \gamma & & \\ & & 1 & \\ & & & 1 \end{pmatrix}.$$

$$\begin{cases} \nu_i\sin\varphi_i = \nu_o\sin\varphi_o \\ \nu_i(-\beta-\cos\varphi_i) = \nu_o(-\beta+\cos\varphi_o) \end{cases}$$

$$\frac{1+\beta}{1-\beta} = \frac{\tan\frac{\varphi_i}{2}}{\tan\frac{\varphi_0}{2}}.$$

$$\sin\varphi_i(-\beta+\cos\varphi_0) = \sin\varphi_0(\beta+\cos\varphi_i)$$

$$X^{\mu\nu} = \left(\quad \right)$$

$$X^{\mu}{}_{\nu} = X^{\mu\alpha} g_{\alpha\nu} = (\bar{\bar{X}}\,\bar{g})^{\mu}{}_{\nu}$$

V^{μ}

$$V_{\mu} X^{\mu\nu} = V^{\mu} X_{\mu}{}^{\nu}$$

守恒 $\partial_\mu T^{\mu\nu}=0$.

$L=\partial^\nu\varphi\,[\partial^2\rho - V'(\varphi)]$.

$T\equiv m^\mu{}_\mu = F^{\mu\lambda}F_{\mu\lambda} - \frac{1}{4}\eta^\mu$

$\mathrm{Tr}(\eta^{\mu\nu})=2$

$\mathrm{Tr}(\eta^\mu{}_\nu)=4$.

Poincare upper plane.

$$ds^2=\frac{dx^2+dy^2}{y^2}$$

$$\Gamma^{\mu}{}_{\nu\lambda} = \frac{1}{2} g^{\mu\sigma} \left(\partial_\lambda g_{\nu\sigma} + \partial_\nu g_{\lambda\sigma} - \partial_\sigma g_{\nu\lambda} \right).$$

$$\begin{cases} \dfrac{d^2 x}{d\lambda^2} - \dfrac{2}{y} \dfrac{dx}{d\lambda} \dfrac{dy}{d\lambda} = 0 \\[2ex] \dfrac{d^2 y}{d\lambda^2} + \dfrac{1}{y} \left(\dfrac{dx}{d\lambda} \right)^2 - \dfrac{1}{y} \left(\dfrac{dy}{d\lambda} \right)^2 = 0. \end{cases}$$

$$\frac{1}{\frac{dx}{d\lambda}} \left(\frac{d^2 x}{d\lambda^2} \right) = \frac{2}{y} \frac{dy}{d\lambda} = 2 \frac{d \ln y}{d\lambda}$$

$$\shortparallel$$

$$\frac{d \ln \left(\frac{dx}{d\lambda} \right)}{d\lambda} = 2 \frac{d \ln y}{d\lambda}$$

$$\ln \left(\frac{dx}{d\lambda} \right) = 2 \ln y + C$$

$$\Rightarrow \quad \frac{dx}{d\lambda} = C y^2$$

$$yy'' - y'^2 + x'^2 = 0$$

$$\Rightarrow \left(\frac{y'}{y}\right)^2 + C^2y^2 = 0$$

$$\frac{d}{d\lambda} = \frac{d}{dx}\frac{dx}{d\lambda} = Cy^2\frac{d}{dx}.$$

$$\left(Cy\frac{dy}{dx}\right)^2 + C^2y^2 = 0.$$

$$\Rightarrow 0 = C^2y^2\left[1 + \frac{d}{dx}\left(y\frac{dy}{dx}\right)\right].$$

$C = 0 \qquad \frac{dx}{d\lambda} = 0.$

$C \neq 0 \quad 1 + \frac{d}{dx}\left(y\frac{dy}{dx}\right) = 0 \therefore$

Chapter 9 3(6). 9(17)

$$\Omega^2 = \frac{GM}{r^3}.$$

$$-c^2 d\tau^2 = ds^2$$