Basic Cosmological Physics

Cosmology.

infrared CCD. binary star / black hole.

Radio Telescope

HI (Atom Hydrogen) 巡天

21 cm HI → Optical Thin.

大尺度星系动力学 (big structure galactic dynamics)

Full-Fisher relation.

SKA.

ALMA - CO 亚毫米波

VLT

Adaptive Optical

Subaru Telescope 昴星团望远镜 8.2 整镜面.

TMT; ELT

HST. (Ultra deep z = 10).

JWST. z = 10 ~ 20.

Fundamental Observer

idealized fluid

substratum.

stationary.

$H_0 \approx 70 km/s/mpc$

$\approx 100 \cdot h \; km/s/mpc \xrightarrow{cas} 3.24 \times 10^{-8} h \cdot s^{-1}$

$h \downarrow 0.7$

inflation

$1/H_0$ Hubble time scale

$\downarrow$

$4.4 \times 10^{17} s \sim 14 Gyr$

Components of Universe

$E^2_{tot} = m_0^2 c^4 + p^2 c^2$

Baryons ~ visible 5%

Neutrinos.

Radiation $E = h\nu = \frac{hc}{\lambda}$

$z \approx 1090$ (CMB)

↓ before

plasma soup.

Dark Matter. no direct observation

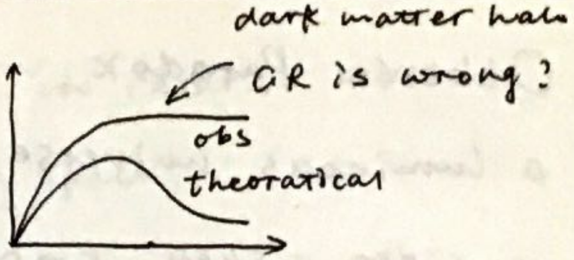
① rotation curves of galaxies.

② Mass to light ratio in clusters

③ Gravitational lensing.

① $G\frac{Mm}{R^2} = m\frac{v^2}{R}$.

$v = \sqrt{\frac{GM}{R}}$



• Dark Energy.

Accelerated Expansion & Cosmological constant

Critical density

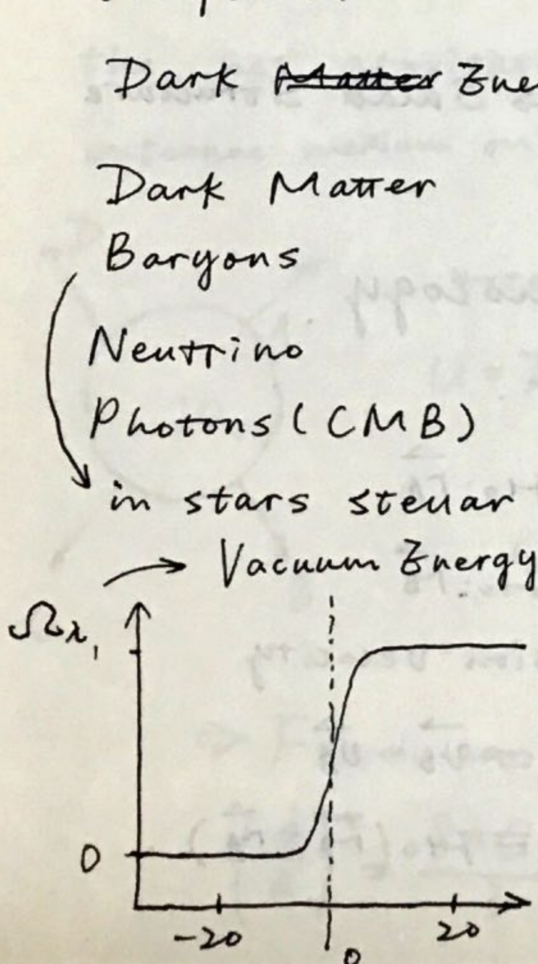
$\rho_c = \frac{3H_0^2}{8\pi G}$ (the universe will expanse for ever, smaller else it will collapse finally).

Cosmic Inventory.

Component	Ω (ρ/ρ_c)
Dark ~~Matter~~ Energy	0.69
Dark Matter	0.31
Baryons	0.049
Neutrino	0.001
Photons (CMB)	5×10^{-5}

in stars stellar 0.003

→ Vacuum Energy / Cosmological Constant.



More fundamental physics?

Olber's Paradox.

a luminous universe?

consider a shell $r \to r+dr$.

stars $n_0 \cdot 4\pi r^2 dr$.

total ~~density~~ intensity of the sky.

$$\mu = \int_0^{r_{max}} 4\pi r^2 n_0 \left(\frac{L_0}{4\pi r^2}\right) dr = n_0 L_0 r_{max}.$$

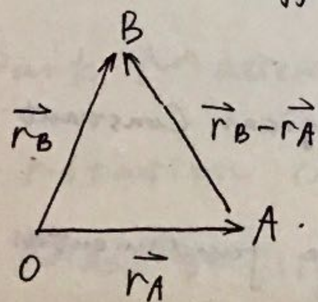
10^{13} times bigger than observation in optical wavelength.

$\Rightarrow$ finite universe.

Neutrino & WDM $\longrightarrow$ Small Structure

Newtonian Cosmology.

GR - Birkhoff's theorem.



$\vec{v}_A = H_0 \cdot \vec{r}_A$

$\vec{v}_B = H_0 \cdot \vec{r}_B$

recession velocity

$$\vec{v}_{BA} = \vec{v}_B - \vec{v}_A$$
$$= H_0(\vec{r}_B - \vec{r}_A).$$

comoving coordinates. no relative motions.

the ruler expanse with space.

comoving distance $\vec{x}_{BA}$.

$\vec{r}_{BA} = a(t)$ $\longrightarrow$ (homogenous & Isotropic)

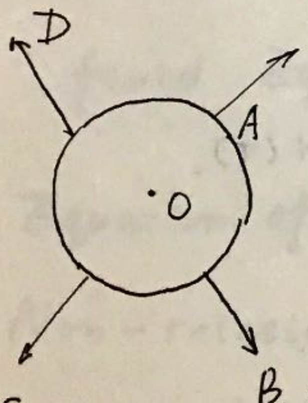
$\downarrow$

physical coordinates. (distance).

so the comoving distance don't change at different redshifts, but physical distance will.

Birkhoff's Theorem.

the net gravitational effect of a uniform external medium on a spherical ~~energy~~ cavity is 0



total energy.

$$U = T + V = \frac{1}{2}m\dot{r}^2 - G\frac{Mm}{r}$$

$$= \frac{1}{2}m\dot{r}^2 - \frac{4\pi}{3}G\rho r^2 m.$$

$$\xlongequal{r=ax} \frac{1}{2}m\dot{a}^2x^2 - \frac{4\pi}{3}G\rho a^2x^2m.$$

$\Rightarrow$ Friedmaan Equation. $k \sim$ curvature.

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G\rho}{3} - \frac{kc^2}{a^2} \qquad kc^2 = -\frac{2U}{mx^2}$$

k should be independent of x.

so $U \propto x^2$

and k is not a function of time.

as U & x are not function of time.

① $k > 0$., $U < 0$. $|V| > T$.

expansion will halt and reverse.

② $k < 0$, $U > 0$ $|V| < T$

expansion for ever

③ $k = 0$, $U = 0$

halt at $t \to \infty$.

$$\vec{v} = H_0 \vec{r} \qquad \vec{v} = \frac{|\dot{\vec{r}}|}{|\vec{r}|} \vec{r} = \frac{\dot{a}}{a} \vec{r}.$$

$$\vec{r} = a\vec{x}$$

Hubble constant $H_0 = \frac{\dot{a}}{a} = H(t)$.

Friedmann Eq.

$$H_0^2 = \frac{8\pi G}{3} \rho - \frac{kc^2}{a^2}$$

critical density let $k = 0$.

$$\rho_c = \frac{3H_0^2}{8\pi G}$$

thermal dynamics

$$dE = TdS - pdV$$

$\downarrow$

mc^2

$\downarrow$

$\frac{4}{3}\pi a^3 \rho c^2$ $\quad \frac{dE}{dt} = 4\pi a^2 \rho c^2 \frac{da}{dt} + \frac{4}{3}\pi a^3 \frac{d\rho}{dt} c^2.$

$$\frac{dV}{dt} = 4\pi a^2 \frac{da}{dt}$$

Assuming a reversible expansion $dS = 0$.

we have.

$$4\pi a^2 \rho c^2 \frac{da}{dt} + \frac{4}{3}\pi a^3 \frac{d\rho}{dt} c^2 + 4\pi a^2 p \frac{da}{dt} = 0.$$

$$\Rightarrow \quad \dot{\rho} + \frac{3\dot{a}}{a}\left(\rho + \frac{p}{c^2}\right) = 0.$$

fluid Equation.

Equation of state. $p \equiv p(\rho)$

Non-relativistic matter. (dust)

$p = 0$ interaction are cause by gravity.

Highly-relativistic particle.

$$p = \frac{u}{3} = \frac{\rho c^2}{3}$$

Friedmann Equation

$$2\frac{\dot a}{a}\frac{a\ddot a-\dot a^2}{a^2}=\frac{8\pi G}{3}\dot\rho+2\frac{kc^2\dot a}{a^3}$$

$\Big\downarrow$ fluid Equation.

$$\frac{\ddot a}{a}-\left(\frac{\dot a}{a}\right)^2=-4\pi G\left(\rho+\frac{p}{c^2}\right)+\frac{kc^2}{a^2}$$

$\Uparrow$

Friedmann Eq again $=\frac{8\pi G}{3}\rho-\frac{kc^2}{a^2}$.

acceleration Equation.

$$\frac{\ddot a}{a}=-\frac{4\pi G}{3}\left(\rho+\frac{3p}{c^2}\right).$$

if ~~we~~ $p>0$, universe expansion velocity decrease.

if we consider cosmological constant Λ, p can be negative.

$k\sim$ curvature

$k=0$ plain

a) pressureless dust

$p=0$. fluid equation.

$$\dot\rho+3\frac{\dot a}{a}\rho=0\Rightarrow\frac{1}{a^3}\frac{d}{dt}(\rho a^3)=0.$$

$$\Rightarrow\rho a^3=\text{const}.$$

$\rho \propto \frac{1}{a^3}$. define a_0 (now) $= 1$

$$\rho = \frac{\rho_0}{a^3}$$ $\rho_0 \rightarrow$ now.

$k=0$. so Friedmann Eq writes as.

$$\dot{a}^2 = \frac{8\pi G \rho_0}{3} \frac{1}{a} \Rightarrow a \propto t^{2/3}$$

$$a(t) = \left(t/t_0\right)^{2/3} \qquad \rho(t) = \frac{\rho_0}{a^3} = \frac{\rho_0 t_0^2}{t^2}$$

* expanse forever

$$H(t) = \frac{\dot{a}}{a} = \frac{2}{3t}$$ decreasing expansion velocity

"Einstein - de Sitter" Cosmology.

$$t_0 = \frac{2}{3} \frac{1}{H_0}$$ $\frac{1}{H_0} \rightarrow$ Hubble timescale.

$t_0 \uparrow$ current age of universe.

$t_0 \approx 9.3$ Gyr. (too young!)

b) $k=0$ radiation.

$$p = \frac{\rho c^2}{3}$$

fluid Eq: $\dot{\rho} + 4\frac{\dot{a}}{a}\rho = 0$.

$$\Rightarrow \rho \propto \frac{1}{a^4}$$

expansion $\downarrow$ wavelength $\uparrow$ $\downarrow$ energy of photon $\downarrow$

$$a(t) = \left(\frac{t}{t_0}\right)^{1/2}$$

$$\rho(t) = \frac{\rho_0}{a^4} = \frac{\rho_0 t_0^2}{t^2}$$

$$H(t) = \frac{\dot{a}}{a} = \frac{1}{2t} < \frac{2}{3t}$$ → dust.

② radiation dominated

$\rho_{rad} \propto \frac{1}{t^2}$ $\quad \rho_{dust} \propto \frac{1}{a^3} \propto \frac{1}{t^{3/2}}$

→ unstable situation.

(radiation maybe dominated at first, but decline fast than dust).

① Dust dominated.

$\rho_{dust} \propto \frac{1}{t^2}$ $\quad \rho_{rad} \propto \frac{1}{t^{8/3}}$

→ stable situation.

Relativistic Cosmology

The Robertson-Walker Metric

3-D space.

measure distance along a curved path P.

$(dl)^2 = (dx)^2 + (dy)^2 + (dz)^2$.

total distance $\Delta l = \int_1^2 dl = \int_{1\,(P)}^2 \sqrt{(dx)^2 + (dy)^2 + (dz)^2}$

a curved world line. ω.

Minkowski space.

$$(ds)^2 = (c\,dt)^2 - (dx)^2 - (dy)^2 - (dz)^2.$$

$$\Delta s = \int_A^B \sqrt{(ds)^2} = \int_A^B \sqrt{(c\,dt)^2 - (dx)^2 - (dy)^2 - (dz)^2}.$$

distance measure between two events A & B.

proper distance $\Delta L = \sqrt{-(\Delta s)^2}$

On the surface of a sphere, curvature is defined as $k \equiv \frac{1}{R^2}$

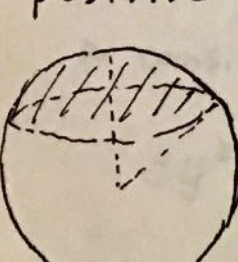
for a more common 2-D case,

$$k = \frac{3}{\pi} \lim_{D \to 0} \frac{2\pi D - C_{measured}}{D^3}.$$

Zero curvature flat geometry.

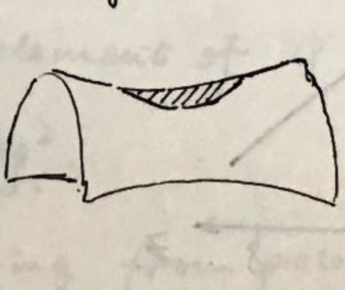
$C = 2\pi D$.

positive curvature.



$C_m < 2\pi D$

negative curvature.



$C_m > 2\pi D$.

Relationship between the Friedmann Eq &

the Einstein's Field Equation (EFE).

$$R_{\alpha\beta} - \frac{1}{2} R g_{\alpha\beta} + \Lambda g_{\alpha\beta} = \frac{8\pi G}{c^4} T_{\alpha\beta}.$$

Geometry of space-time produced by a given given distribution of mass & energy.

Einstein's Tensor.

$$G_{\alpha\beta} = R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R.$$

$R_{\alpha\beta}$: Ricci tensor

$$ds^2 = \underbrace{g_{\alpha\beta}}_{\text{Metric Tensor}} dx^\alpha dx_\beta.$$

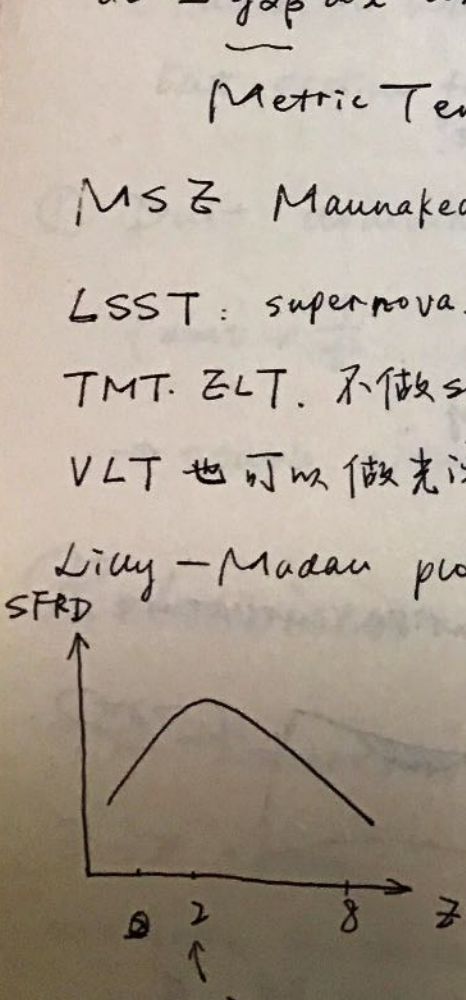
MSE Maunakea spectroscopic Explorer. 仪器在主焦点.

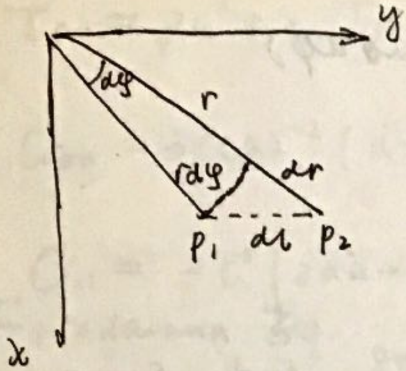
LSST: supernova. imaging.

TMT. ELT. 不做survey. small FOV.

VLT 也可以做光谱巡天.

Lilly-Madau plot





球面上两点距离.

distance between 2 points P_1, P_2.

$$dl^2 = \cancel{dR^2 +} R^2 \cancel{d\varphi}\, d\theta^2 + R^2 \sin^2\theta \, d\varphi^2.$$

def $r = R\cos\theta$.

$$R\,d\theta = \frac{dr}{\cos\theta} = \frac{R\,dr}{\sqrt{R^2 - r^2}}$$

$$= \frac{dr}{\sqrt{1-\left(\frac{r}{R}\right)^2}}$$

$$\Rightarrow (dl)^2 = \left(\frac{dr}{\sqrt{1 - r^2/R^2}}\right)^2 + (r\,d\phi)^2.$$

$$K(\text{curvature}) = \frac{1}{R^2}.$$

if $K = 0$, $R \to \infty$. $(dl)^2 = (dr)^2 + (r\,d\phi)^2$

(2D + curvature → 3D).

so what about 3D + curvature?

consider angular element of φ.

$$d\varphi^2 = d\theta^2 + \sin^2\theta\, d\varphi^2.$$

extend to 3D by changing from polar to spherical coordinate.

$$(dl)^2 = \left(\frac{dr}{\sqrt{1-kr^2}}\right)^2 + (rd\theta)^2 + (r\sin\theta\, d\phi)^2.$$

if we add time.

$$(ds)^2 = (dt)^2 - (dl)^2.$$

proper distance:

$dt = 0.$ $\Delta L = \sqrt{-(\Delta s)^2}$ ~~with~~

consider $r(t) = a(t) \cdot x$ → comoving coordinates.

↓

radial coordinates

let $K(t) = \frac{k}{a^2(t)}$.

Robertson-Walker metric.

$$(ds)^2 = (cdt)^2 - a^2(t)\left[\left(\frac{dr}{\sqrt{1-kr^2}}\right)^2 + (rd\theta)^2 + (r\sin\theta d\phi)^2\right]$$

here $r \to$ comoving distance.

Friedmann Eq.

EFE $R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R = \frac{8\pi G}{c^4} T_{\alpha\beta}.$

metric $ds^2 = g_{\alpha\beta} dx^\alpha dx^\beta$

$g_{00} = 1$ $g_{11} = -\frac{a^2}{1-kr^2}$ $g_{22} = -a^2 r^2$ $g_{33} = -a^2 \sin^2\theta\, r^2.$

$T_{00} = \rho c^2$. $\quad T_{11} = \frac{p a^2}{1-kr^2}$

$G_{00} = 3(ca)^{-2}(\dot{a}+kc)^2$

$G_{11} = -c^{-2}(2a\ddot{a}+\dot{a}^2+k)(1-kr^2)^{-1}$

Friedmann Eq

$$\begin{cases} \left(\frac{\dot{a}}{a}\right)^2 + \frac{kc^2}{a^2} = \frac{8\pi}{3}G\rho. \\ 2\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2 + \frac{kc^2}{a^2} = -\frac{8\pi}{c^2}Gp. \end{cases}$$

① + ② ⇒ Acceleration equation

$$\frac{\ddot{a}}{a} = -\frac{4\pi}{3c^2}G(\rho c^2 + 3p).$$

critical density $\rho_c = \frac{3H_0^2}{8\pi G}$, $H_0 = \frac{\dot{a}}{a}$

at present time t_0. def $\Omega_0 = \frac{\rho_0}{\rho_c}$

rewrite Friedmann Eq as.

$$\dot{a}_0^2 = \frac{8\pi G}{3}a_0^2\rho_0 - kc^2$$

$$= H_0^2 a_0^2 \Omega_0 - kc^2$$

$$\Rightarrow H_0^2 a_0^2 = H_0^2 a_0^2 \Omega_0 - kc^2$$

$$kc^2 = H_0^2 a_0^2(\Omega_0 - 1)$$

$$k = +1,\ 0,\ -1$$

$$\begin{cases} \Omega_0 > 1 \\ \Omega_0 = 1 \\ 0 < \Omega_0 < 1 \end{cases}$$

Difference between NM

$$\begin{cases} kc^2 = -\frac{2U}{mx^2} \\ K(t) = \frac{k}{a^2(t)}. \end{cases}$$

Cosmological Constant Λ

1929 Hubble found the universe is expansing.

However, people used to believed the universe is static. so

$$a \neq f(t) \quad \dot{a} = \ddot{a} = 0. \quad H_0 = \frac{\dot{a}}{a} = 0.$$

$$\text{age of universe } t_0 \sim \frac{1}{H_0} \approx \infty$$

in Friedmann Eq, let $\dot{a} = \ddot{a} = 0$. we have

$$k \frac{c^2}{a^2} = \frac{8\pi}{3} G \rho_0 = -\frac{8\pi}{c^2} G p_0.$$

$$\rho_0 > 0, \ k > 0 \Rightarrow p_0 < 0 \ ??$$

to solve this problem. Einstein add a term

in EFE. $G_{\alpha\beta} - \Lambda g_{\alpha\beta} = \frac{8\pi G}{c^4} T_{\alpha\beta}.$

Friedmann Eq writes:

$$\left(\frac{\dot{a}}{a}\right)^2 + \frac{kc^2}{a^2} = \frac{8\pi}{3} G\rho + \frac{\Lambda c^2}{3}$$

$$2\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2 + \frac{kc^2}{a^2} = -\frac{8\pi}{c^2} Gp + \Lambda c^2$$

Acceleration Eq. & Friedmann Eq 1

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3c^2}\left(\rho c^2 + 3p\right) + \frac{1}{3}\Lambda c^2$$

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\left(\rho + \frac{\Lambda c^2}{8\pi G}\right) - k\frac{c^2}{a^2}.$$

we define vacuum energy density.

$$\rho_{vac} = \frac{\Lambda c^2}{8\pi G} \rightarrow \Lambda = \frac{8\pi G \rho_{vac}}{c^2}$$

Friedmann Eq 2.

$$2\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2 + \frac{kc^2}{a^2} = -\frac{8\pi G}{c^2}\left(p - \frac{\Lambda c^4}{8\pi G}\right)$$

if $\Lambda < 0$. $\ddot{a} < 0$. $\rightarrow$ de·

in Newtonian cosmology.

add an additional potential energy.

$$V_\Lambda \equiv -\frac{1}{6}\Lambda m c^2 r^2.$$

$$U = T + V + V_\Lambda$$
$$= \frac{1}{2} m \dot{r}^2 - \frac{4}{3}\pi G \rho r^2 m - \frac{1}{6}\Lambda m c^2 r^2.$$

$$F_\Lambda = -\frac{\partial V_\Lambda}{\partial r}\hat{r} = \frac{1}{3}\Lambda m c^2 \hat{r}$$

World Models. we want to know the variance of $a(t)$.

natural units. $c = 1$ $E = mc^2$.

$H(t) = \frac{\dot{a}}{a}$. $\rho_\Lambda \equiv \frac{\Lambda}{8\pi G}$

Friedmann Eq writes.

$$H^2 + \frac{k}{a^2} = \frac{8\pi G}{3}\left(\sum_i \rho_i + \rho_\Lambda\right).$$

ρ_i, matter, radiation.

$k=0.$ $\rho_{tot} \equiv \sum_i \rho_i + \rho_\Lambda = \frac{3H^2}{8\pi G} \equiv \rho_{critic}$

fraction of the critical density contributed by each component of the universe.

$$\Omega_i \equiv \frac{\rho_i}{\rho_{tot}}$$

$\Omega_m(t)$ $\Omega_r(t)$ $\Omega_\Lambda(t)$.

matter radiation dark energy.

$\cancel{\Omega_m}$

0.3 ~~0.7~~ 0.7

Friedmann Eq writes.

$$\frac{k}{a^2H^2} = \sum_i \Omega_i + \Omega_\Lambda - 1$$

def $\frac{k}{a^2H^2} = -\Omega_K.$

$$\Rightarrow \sum_i \Omega_i + \Omega_\Lambda + \Omega_K = 1$$

assume $p=0.$, pressure less matter.

i) Flat FRW Cosmologies.

$$\underset{\downarrow}{k=0} \quad \Omega_K = 0.$$

$\rho_m = \rho_{m,0}\left(\frac{a}{a_0}\right)^{-3}$, let $a_0 \equiv 1$

$$\left(\frac{\dot a}{a}\right)^2 = \frac{8\pi G}{3}\rho_{m,0} + \frac{\Lambda}{3}$$

$$\Rightarrow \dot{a}^2 = H_0^2 \Omega_{m,0} a^{-1} + H_0^2 \Omega_{\Lambda,0} a^2$$

no radiation $\Rightarrow \Omega_{m,0} + \Omega_{\Lambda,0} = 1$.

① $\Lambda > 0$,

let $u = \dfrac{2\Omega_{\Lambda,0}}{\Omega_{m,0}} a^3$

~~the eq writes as:~~ we get:

$$\dot{u}^2 = 9H_0^2 \Omega_{\Lambda,0} [2u + u^2] = 3\Lambda [2u + u^2].$$

$$\int_0^u \frac{du}{(2u+u^2)^{1/2}} = \int_0^t (3\Lambda)^{1/2} dt = (3\Lambda)^{1/2} t.$$

let $v = u + 1$, $\cosh w = v$.

we get $\displaystyle\int_0^u \frac{du}{(v^2-1)^{1/2}} = \int_1^v \frac{dv}{(v^2-1)^{1/2}} = \omega.$

$$\Rightarrow a^3 = \frac{\Omega_{m,0}}{2\Omega_{\Lambda,0}} \left[\cosh (3\Lambda)^{1/2} t - 1\right].$$

② $\Lambda < 0$.

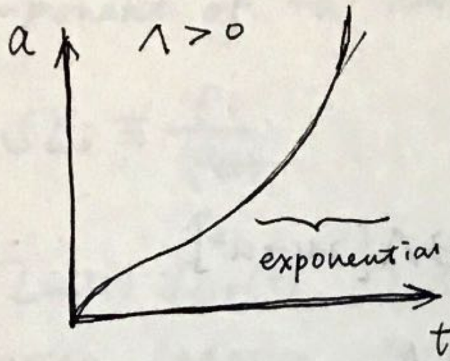
let. $u = -\dfrac{2\Omega_{\Lambda,0}}{\Omega_{m,0}} a^3$.

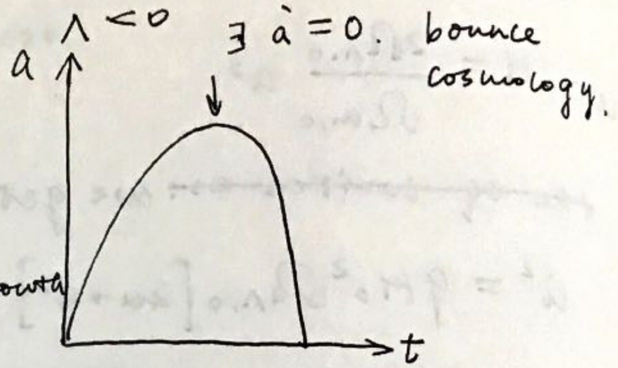
we get

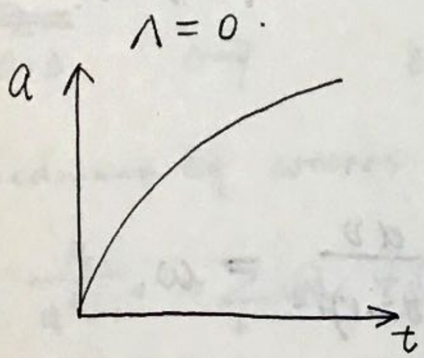
$$a^3 = \frac{\Omega_{m,0}}{-2\Omega_{\Lambda,0}} \left\{ 1 - \cos [3(-\Lambda)]^{1/2} t \right\}.$$

③ $\Lambda = 0 \Rightarrow$ Einstein de Sitter Cosmology.

$$\left.\begin{aligned} a &= \left(\frac{t}{t_0}\right)^{2/3} \\ t_0 &= \frac{2}{3H_0} \end{aligned}\right\} \longrightarrow a = \left(\frac{9}{4} H_0^2 t^2\right)^{1/3}.$$







from $\dot a^2 = H_0^2 \Omega_{m,0} a^{-1} + H_0^2 \Omega_{\Lambda,0} a^2$

$\Lambda > 0$.

$$\left(\frac{\dot a}{a}\right)^2 = \frac{8\pi G}{3} \rho_{m,0} a^{-3} + \frac{\Lambda}{3}$$

small a $\Rightarrow \dot a / a \propto \Lambda/3$.

$$a \sim e^{\left(\frac{\Lambda}{3}\right)^{1/2} t}$$

$\Lambda < 0$. let $\dot a = 0 \Rightarrow a^{-3} = - \frac{3H_0^2}{8\pi G} \frac{\Omega_{\Lambda,0}}{\rho_{m,0}} = -\frac{\rho_c}{\rho_{m,0}} \Omega_{\Lambda}$

$\Rightarrow a = \left(-\frac{\Omega_{m,0}}{\Omega_{\Lambda,0}}\right)^{1/3}$: the maximum scale factor.

ii) Cosmologies with $k \neq 0$ and $\Lambda = 0$. no radiation.

$$\dot{a}^2 = \Omega_{m,0} H_0^2 a^{-1} - k = \Omega_{m,0} H_0^2 a^{-1} + \Omega_{k,0} H_0^2$$

$\Omega_{k,0} = 1 - \Omega_{m,0}$.

$\Omega_{k,0} > 0$ $k < 0$. $\Omega_k \equiv -\frac{k}{(aH)^2}$.

when a is large. $\dot{a} \simeq \Omega_{k,0} H_0^2 = -k > 0$.

$a \propto t$. $\rightarrow$ a grows linearly with time.

$\Omega_{k,0} < 0$ $k > 0$. positive curvature.

at some time $\dot{a} = 0$.

$\Omega_{m,0} H_0^2 a^{-1} + \Omega_{k,0} H_0^2 = 0$

$$\Rightarrow a_{max} = \frac{\Omega_{m,0} \cancel{H_0}}{|\Omega_{k,0}|}$$

Analytical Soution.

let $u^2 = -\frac{a}{a_{max}} = a\frac{k}{\Omega_{m,0} H_0^2}$

$$\dot{u}^2 = \frac{u^{-2} H_0^2 |\Omega_{k,0}|^3}{4 \Omega_{m,0}^2}\left[u^{-2} - 1\right]$$

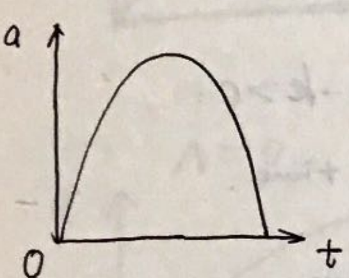
let $u = \sin\theta$.

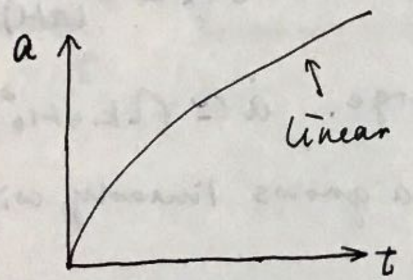
$$t = C_1 \left\{ \sin^{-1}\left(\frac{a}{a_{max}}\right)^{1/2} - \left(\frac{a}{a_{max}}\right)^{1/2}\left(1 - \frac{a}{a_{max}}\right)^{1/2} \right\}.$$

$$C_1 = \frac{\Omega_{m,0}}{|\Omega_{k,0}|^{3/2} H_0}$$

$k<0$, $\Omega_{k,0} > 0$.

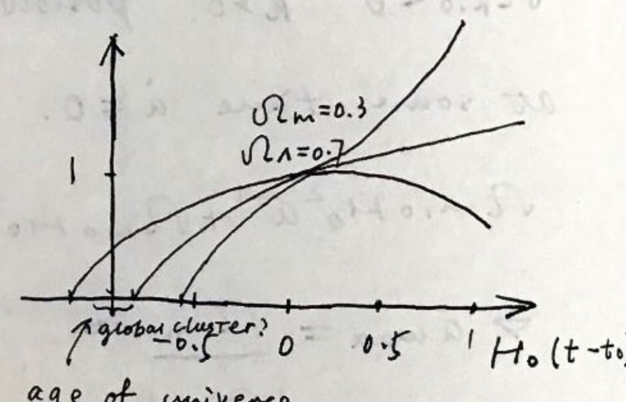
$$t = C_1 \left\{ -\sinh^{-1}\left(\frac{a}{a_{max}}\right)^{1/2} + \left(\frac{a}{a_{max}}\right)^{1/2}\left(1 + \frac{a}{a_{max}}\right)^{1/2} \right\}.$$





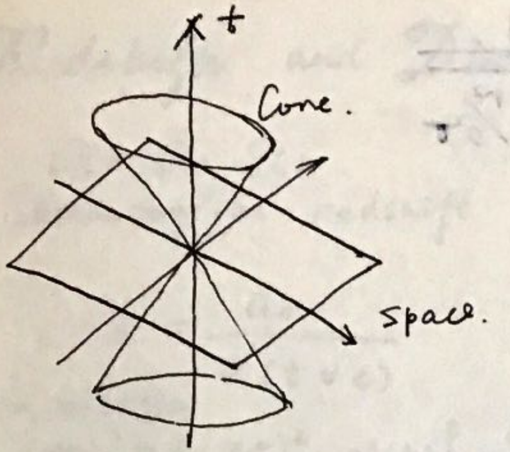
$k=0$, $\rho = 0$. ($\Omega_{m,0} = 0$)

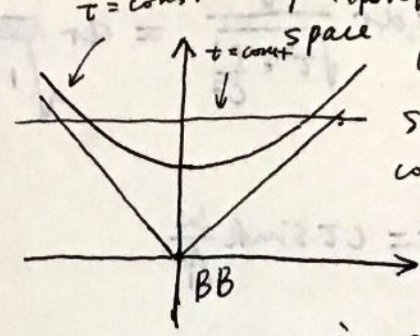
$$\left(\frac{\dot a}{a}\right)^2 = \frac{\Lambda}{3} \Rightarrow a = e^{\left(\frac{\Lambda}{3}\right)^{1/2} t}$$



A Special Relativistic Model
the Milne Universe

empty universe; no gravitational force.





Minkowski space
(t, r, θ, ϕ).
spherical coordinates.

radius velocity only. → in a comoving coordinates system. of a particle from big bang

$(\tau, \omega, \theta, \phi)$

proper time $\tau = \left(t - \frac{vr}{c^2}\right)\gamma$.

$= t\sqrt{1 - \frac{r^2}{c^2t^2}}$ ← time dilation.

line element ds

$= t\sqrt{1 - \frac{v^2}{c^2}}$

$$ds^2 = c^2dt^2 - dr^2 - r^2 d\Omega^2$$

only consider radius direction.

$$ds^2 = c^2dt^2 - dr^2$$

inv with coordinate

$$= c^2d\tau^2 - dl^2.$$

τ = const surface. $d\tau = 0$.

$$\Rightarrow dl^2 = dr^2 - c^2dt^2$$

$$= dr^2\left(1 - \frac{c^2dt^2}{dr^2}\right)$$

$$= dr^2\left(1 - \frac{r^2}{c^2t^2}\right) = dr^2\left(1 - \frac{v^2}{c^2}\right) = dr^2\frac{\tau^2}{t^2}$$

$$\Rightarrow dl = dr \frac{\tau}{\sqrt{\tau^2 + \frac{r^2}{c^2}}} = dr \frac{1}{\sqrt{1 + r^2/c^2\tau^2}}$$

let $r = c\tau \sinh \frac{\omega}{A}$

$$dr = \frac{c\tau}{A} \cosh \frac{\omega}{A} \, d\omega.$$

$$dl = \frac{c\tau}{A} d\omega.$$

$\tau = \text{const}$ surface (~~angu~~ 切向也加上).

$$ds^2 = -dl^2 - r^2 d\Omega^2.$$

$$= -\left(\frac{c\tau}{A}\right)^2 d\omega^2 - (c\tau)^2 \sinh^2 \frac{\omega}{A} d\Omega^2$$

$\Downarrow$

general form.

$$ds^2 = c^2 dt^2 - \left(\frac{c\tau}{A}\right)^2 \left[d\omega^2 + A^2 \sinh^2 \frac{\omega}{A} d\Omega^2 \right].$$

in Robertson Walker metric. let $k = -1$. $a(t) = \frac{c\tau}{A}$.

(increase linearly with time).

$A = \frac{c}{\dot{a}}$ curvature

Redshifts and Distance

$\Omega_i, \Omega_\Lambda, \Omega_k$

Cosmological redshift

$$1+z = \frac{a_0}{a(t=e)}.$$

RW metric

$$(ds)^2 = (c\,dt)^2 - a^2(t)\left[\frac{dr^2}{1-kr^2} + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)\right]$$

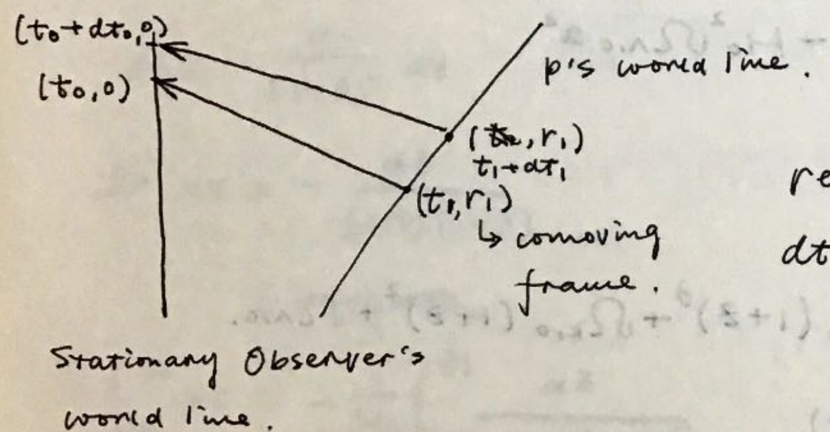
(Here r is comoving distance)

GR. propogation of light $ds = 0$.

Let observer $r = 0$, $d\phi = d\theta = 0$

RW metric $\rightarrow$ ~~ds^2~~ $\quad \dfrac{c\,dt}{a(t)} = \pm \dfrac{dr}{(1-kr^2)^{1/2}}$

consider two arbitary world lines.



relations between dt_1 & dt_0?

integrate.

$$\int_{t_e}^{t_0} \frac{dt}{a(t)} = -\frac{1}{c}\int_{r_e}^{0} \frac{dr}{\sqrt{1-kr^2}}$$

$$\int_{t_e+\Delta t_e}^{t_0} \frac{dt}{a(t)} = -\frac{1}{c}\int_{r_e}^{0} \frac{dr}{\sqrt{1-kr^2}}$$

$$\Rightarrow \int_{t_e+\Delta t_e}^{t_0+\Delta t_0} \frac{dt}{a(t)} - \int_{t_e}^{t_0} \frac{dt}{a(t)} = 0.$$

$$\Rightarrow -\int_{t_e}^{t_e+\Delta t_e} \frac{dt}{a(t)} + \int_{t_0}^{t_0+\Delta t_0} \frac{dt}{a(t)} = 0.$$

$$\frac{\Delta t_e}{a(t_e)} = \frac{\Delta t_0}{a(t_0)}.$$

$$\Rightarrow \frac{\Delta t_e}{\Delta t_0} = \frac{a(t_e)}{a(t_0)}$$

$$\frac{\lambda_0}{\lambda_e} = \frac{a(t_0)}{a(t_e)} = 1+z$$

$\dot{H}(z)$ =

from Friedmann Eq.

$$\dot{a}^2 = H_0^2 \Omega_{m,0} a^{-1} + H_0^2 \Omega_{\Lambda,0} a^2.$$

$$\Omega_{k,0} \equiv -\frac{k^2}{(aH)^2}$$

$$\left(\frac{H(z)}{H_0}\right)^2 = \Omega_{m,0}(1+z)^3 + \Omega_{k,0}(1+z)^2 + \Omega_{\Lambda,0}.$$

Define RHS $= E(z)$.

$$\Rightarrow H(z) = H_0 \sqrt{E(z)}$$

i) Einstein de Sitter Cosmology.

$$\Omega_{m,0} = 1, \quad \Omega_{k,0} = \Omega_{\Lambda,0} = 0.$$

$$H(z) = H_0 (1+z)^{3/2}.$$

consensus cosmology. $\Omega_{m,0} \sim 0.3$. $\Omega_k = 0$ $\Omega_\Lambda \sim 0.7$.

$$H(z) = H_0 \sqrt{\Omega_{m,0}(1+z)^3 + \Omega_{\Lambda,0}}$$

if we take radiation into account,

$$E'(z) = E(z) + \Omega_{rad}(1+z)^4$$

↑ important at high redshift.

dark energy dominates the expansion after $z < 0.3 \sim 0.5$.

Redshift vs. time.

$$H(z) \equiv \frac{\dot a}{a} = \frac{da}{dz} \cdot \frac{dz}{dt} \frac{1}{a} = \frac{da}{dz} \cdot \frac{dz}{dt} \cdot \frac{1+z}{a_0}$$

$$da = -\frac{a_0}{(1+z)^2} dz.$$

$$\Rightarrow dt = -\frac{dz}{H(z)(1+z)}$$

↓ integrate.

$$\int_{t_1}^{t_2} dt = -\frac{1}{H_0} \int_{z_1}^{z_2} \frac{dz}{(1+z)(E(z))^{1/2}}$$

age of universe.

$$t_0 = \int_0^{t_0} dt = \underbrace{\frac{1}{H_0}}_{14\,Gyr} \int_0^\infty \frac{dz}{(1+z)\, E(z)^{1/2}}.$$

consensus cosmology $= 13.8$ Gyr.

for Einstein de Sitter cosmology.

$\Omega_{m,0} = 1 \quad \Omega_{k,0} = \Omega_{\Lambda,0} = 0.$

$$t_0 = \frac{1}{H_0} \int_0^{\infty} \frac{dz}{(1+z)^{5/2}} = \frac{2}{3} \frac{1}{H_0}$$

proper distance.

distance between two events A & B in a reference frame

$t_A = t_B.$

$$(ds)^2 = (c\,dt)^2 - a^2(t) \left[\frac{dr^2}{1-kr^2} + r^2 (d\theta^2 + \sin^2\theta\, d\phi^2) \right].$$

let $d\theta = d\phi = 0, \quad dt = 0.$

proper distance

$$s(t) = \int_0^s ds' = \int_0^r a(t) \frac{dr}{\sqrt{1-kr^2}}$$

$$= \begin{cases} \frac{a(t)}{\sqrt{k}} \arcsin(\sqrt{k}\, r). & k > 0 \\ a(t)\, r & k = 0 \\ \frac{a(t)}{\sqrt{k}} \operatorname{arcsinh}(\sqrt{k}\, r) & k < 0. \end{cases}$$

in flat universe proper distance $= a(t) \cdot r$.

closed universe $k > 0$. proper distance $> r$.

open universe $k < 0$ proper distance $< r$.

Horizon

particle Horizon: proper distance to the furthest observable point at time t.

noted as $S_h(t)$.

if $d > S_h(t) \Rightarrow$ out of causal link

for photon $ds = 0$. $d\phi = 0 = d\theta$.

$$\int_0^t \frac{dt}{a(t)} = \frac{1}{c}\int_0^{r_{hor}} \frac{dr}{(1-kr^2)^{1/2}}$$

$$\Rightarrow r_{hor} = \begin{cases} \sin\left(c\int_0^t \frac{dt}{a(t)}\right) & k=1 \\ c\int_0^t \frac{dt}{a(t)} & k=0. \\ \sinh\left(c\int_0^t \frac{dt}{a(t)}\right) & k=-1 \end{cases}$$

for dust / radiation dominant universe can get ~~explicit~~ solution.
explicit

proper distance to r_{hor}

$$S_{hor}(t) = \int_0^{r_{hor}} \frac{dr}{(1-kr^2)^{1/2}}.$$

$$= a(t)\int_0^{r_{hor}} \frac{c\,dt}{a(t)}$$

Riadiation – Dominant $a \propto t^{1/2}$

$S_h = 2ct$.

Dust – Dominant $a \propto t^{2/3}$

$S_h = 3ct$

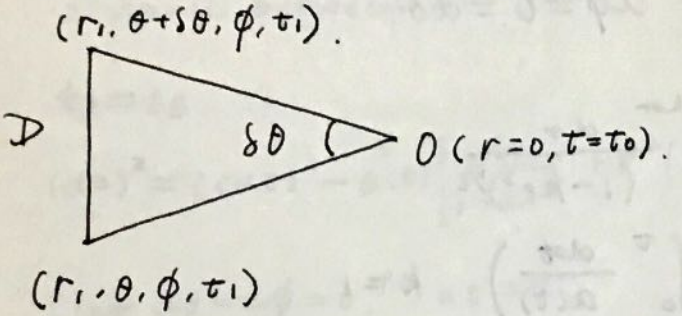
Event Horizon

sets the limit on communication to the future.

limit of integration $t \to t_{max}$ (∞ for open universe

limited value for closed universe)

Angular Diameter Distance



proper distance between the two ends of the object $D = a(t_1) r \delta\theta$

angular diameter $\delta\theta = \dfrac{D}{a(t_1) r}$

in Euclidean Geometry $\delta\theta = \dfrac{D}{d}$

so we define angular diameter distance as:

$$d_A \equiv \frac{D}{\delta\theta} = a_1(t_1) r_1 = \frac{r_1}{1+z}$$

$$\int_{t_1}^{t_0} \frac{dt}{a(t)} = c\int_0^z \frac{dz}{\cancel{H_0}H(z)} = \frac{1}{\sqrt{|k|}} S_k^{-1}(|k|^{1/2} r_1).$$

$$S_k \equiv \begin{cases} \sin(x) & k>0 \\ x & k=0 \\ \sinh(x) & k<0. \end{cases}$$

(Here we use $\dfrac{dz}{dt} = -\dfrac{\dot{a}}{a^2} = -\dfrac{H}{a}$)

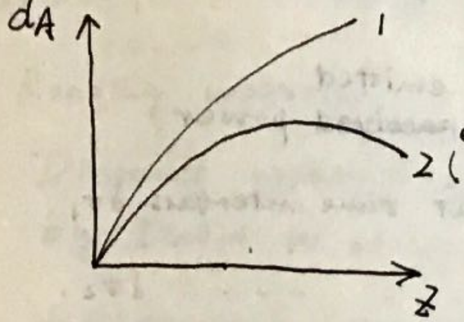
as $k = -aH^2\Omega_{k,0}$.

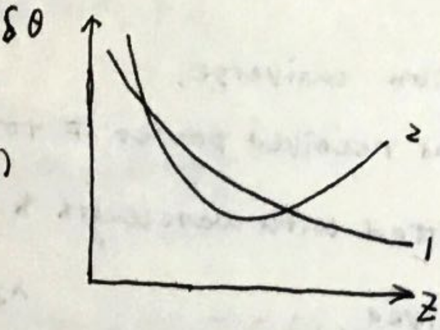
$$d_A(z) = \frac{c}{\sqrt{|\Omega_{k,0}|}\, H_0 (1+z)} \cdot S_k\left(H_0 \sqrt{|\Omega_{k,0}|} \int_0^z \frac{dz}{H(z)} \right).$$

$\Omega_k = 0$. today's consensus cosmology.

$$d_A(z) = \frac{c}{H_0} \frac{1}{1+z} \int_0^z \frac{dz}{\left(\Omega_{m,0}(1+k)^3 + \Omega_{\Lambda,0}\right)^{1/2}}$$

in some cosmology models. (like consensus cosmology)





in Einstein de Sitter cosmology

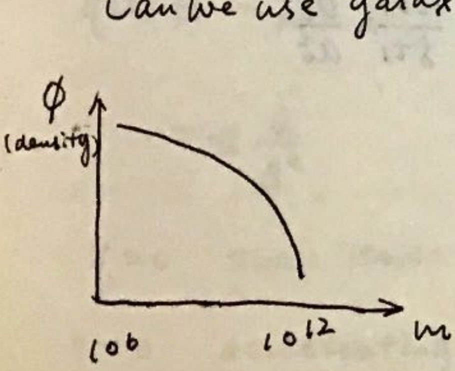
$\Omega_{m,0} = 1$, $\Omega_{\Lambda,0} = \Omega_{k,0} = 0$. $z_{max} = 1.25$.

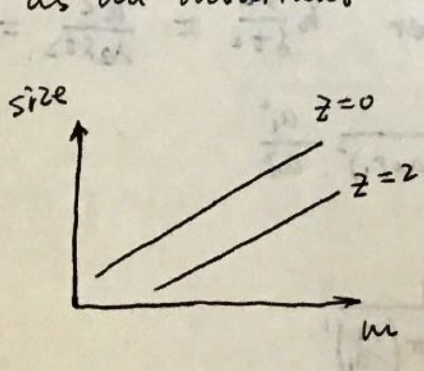
$\delta\theta_{min} = \delta\theta(z_{max}) = 3.375 \frac{H_0 D}{c}$

$D = 1$ Mpc. typical cluster.

$\delta\theta_{min} \approx 4h$.

Can we use galaxy as an (standard) invariant observing object?





CMB map ?

Luminosity Distance d_L

observed flux $F_{obs} = \frac{L}{4\pi d_L^2}$ L : absolute luminosity / total power.

↓

bolometric luminosity (hard to observe in every passband).

Expansion universe.

total received power $\neq$ total ~~received~~ emitted power.

$\Rightarrow$ $\begin{cases} \text{emitted with wavelength } \delta\lambda_1 \text{ at time intervals } \delta t_1 \\ \neq \text{received} \quad - - - \quad \lambda_2 \quad - - - \quad - - - \quad \delta t_2. \end{cases}$

$$\frac{\lambda_1}{\lambda_0} = \frac{a_1}{a_0}$$

a single proton.

$$h\nu = \frac{hc}{\lambda}.$$

Emitted power $h\frac{\nu_1}{\delta t_1} = \frac{hc}{\lambda_1 \delta t_1}$ =

received power $h\frac{\nu_0}{\delta t_2} = \frac{hc}{\lambda_0 \delta t_2} = \frac{h\nu_1}{\delta t_1} \cdot \frac{a_1^2}{a_0^2}$

$$F_{obs} = L \cdot \frac{1}{4\pi(a_0 r_1)^2} \cdot \frac{a_1^2}{a_0^2}$$

let $a_0 = 1$

$$F_{obs} = L \cdot \frac{1}{4\pi\left(r_1/a_1\right)^2}$$

$$\Rightarrow d_L = \frac{r_1}{a_1} = (1+z)\, r_1$$

remember $d_A = \frac{r_1}{1+z}$

$$\Rightarrow d_L = (1+z)^2 d_A.$$

$$= \frac{c(1+z)}{\sqrt{|\Omega_{k,0}|} H_0} \cdot S_k\left(H_0 \sqrt{|\Omega_{k,0}|} \int_0^z \frac{dz}{H(z)} \right).$$

The deceleration Parameter

when $z \ll 1$. $d_p \simeq d_A \simeq d_L \simeq L_1$

Reading material.

Distance measures in cosmology.

by David. W. Hogg.

for small z. expand $E(z) = \Omega_{m,0}(1+3z) + (1-\Omega_{m,0}-\Omega_{\Lambda,0})$

$(1+2z) + \Omega_{\Lambda,0}$.

$$\Rightarrow E(z) = 1 + 2z\left(\frac{1}{2}\Omega_{m,0} - \Omega_{\Lambda,0} + 1\right).$$

def $q_0 = \frac{1}{2}\Omega_{m,0} - \Omega_{\Lambda,0}$

$$E(z) = 1 + 2z(q_0 + 1)$$

$$q(t) \equiv -\frac{1}{H^2}\frac{\ddot{a}}{a}$$

$$= -a\frac{\ddot{a}}{\dot{a}^2}$$

$q > 0$ slow down

$q < 0$ accelerating.

$$\int_0^z \frac{dz}{H(z)} = \frac{1}{H_0}\int_0^z \frac{dz}{E(z)}$$

$$\approx \frac{1}{H_0}\left(z - (q_0+1)\frac{z^2}{2}\right)$$

$$d_L = c(1+z)\, r_1$$

Cepheid variable.

$$= \underbrace{\frac{c}{H_0}}_{\text{(可以用来测}H_0)}\left(z + \frac{1}{2}(1-q_0)z^2 + \cdots\right).$$

monochromatic flux?

emitted frequency ν_e $\qquad d\nu_0 = \frac{d\nu_e}{1+z}$

observed frequency ν_0.

observed bandwidth $\Delta\nu_0$ corresponds to a greater bandwidth in the emitted frame $\Delta\nu_e$.

we have a fixed observation band.

$\Rightarrow$ bandwidth stretching term:

increased the observed flux density of $(1+z)$.

$$F_\nu(\nu) = \frac{L_\nu(\nu(1+z))}{4\pi d_L^2}\cdot(1+z).$$

$$= \frac{L_\nu(\nu(1+z))}{4\pi r_1^2(1+z)}.$$

monochromatic flux density per wavelength interval.

$$d\lambda_0 = d\lambda_e(1+z).$$

$$f_\lambda(\lambda) = \frac{L_\lambda(\lambda/1+z)}{4\pi r_1^2(1+z)^3}$$

K - correction

distance modulus

$$DM = 5\log\frac{d_L}{10pc} = 5\log\frac{r(1+z)}{10pc}.$$

define a wavelength - dependent K - correction. as.

$$K(z,\nu) = -2.5\log\left[\frac{(1+z)L_\nu(\nu(1+z))}{L_\nu(\nu)}\right]$$

$$= -2.5\log\left[\frac{L_\lambda(\lambda/1+z)}{(1+z)L_\lambda(\lambda)}\right]$$

apparent magnitude m of a source with absolute magnitude M. at z at a frequency ν.

$$m(\nu,z) = M(\nu) + DM + K(z,\nu).$$

~~→~~ larger red shift (observation band → infrared).

Surface Brightness

flux density per unit angular area of a spatially extended object.

$$\mu = \frac{f}{\pi\theta^2} \leftarrow \theta = \frac{D}{d_A}$$

$$= \frac{f d_A^2}{\pi D^2} \leftarrow d_L = (1+z)^2 d_A$$

$$= \frac{f d_L^2}{\pi(1+z)^4 D^2}$$

bolometric flux

$$f = \frac{L}{4\pi d_L^2}$$

inv

↑

$$\mu = \frac{L}{4\pi D^2}\cdot\frac{1}{(1+z)^4}. \quad \mu(z) = \frac{\mu(z=0)}{(1+z)^4}$$

largely decreased ↑ (long exposure time).

mono~~matic~~chromatic.

$$\mu_{\nu.obs}(\nu) = \frac{\mu_{\nu.em}(\nu(1+z))}{(1+z)^3}$$

$$\mu_{\lambda.obs}(\lambda) = \frac{\mu_{\lambda.em}(\lambda/(1+z))}{(1+z)^5}$$

dimming (derived from RW metric).

↓

Tolman test.

Black-body Radiation

emitted surface brightness.

$$I_{\nu.em} = \frac{2h\nu^3}{c^2}\frac{1}{e^{h\nu/k_BT}-1}$$

observed:

$$I_{o.em} = \frac{2h(\nu(1+z))^3}{c^2}\frac{1}{e^{h\nu(1+z)/k_BT}-1} \Big/ (1+z)^3$$

$$= \frac{2h\nu^3}{c^2}\frac{1}{e^{h\nu(1+z)/k_BT}-1}$$

a black-body spectrum $T_{obs} = T/1+z$

CMB $z = ?$

$T_{CMB.0} = 2.725K$.

temperature when photons decouple from baryons (~~esc~~ escape from plasma soup) $T \approx 3000K$ $z = T/T_{CMB.0} \approx 1090$

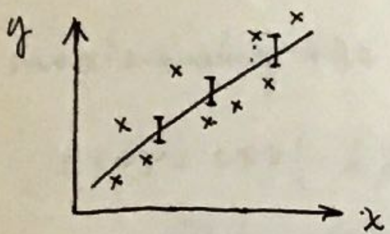
Observation

$$F_{obs} = \frac{L}{4\pi d_L^2}$$

$d_L = f(z)$.

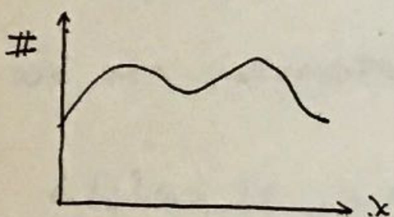
Luminosity Function (LF).

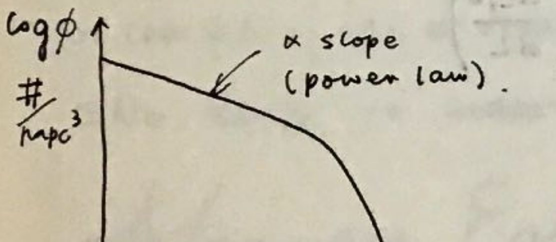
① Scaling Relation.



$y \sim bx$

② Distribution Function.





limitations of observation

① selection effect

$z \uparrow L \uparrow$ (larger star formation rate)

$$\phi(L)dL = \phi^* \left(\frac{L}{L_*}\right)^{\alpha} e^{-L/L^*} \frac{dL}{L^*}$$

Schechter Function.

$\alpha < -2$ $\int_0^\infty \phi(L)dL$ diverge.

observation $\alpha \approx -1.5$

how to get m^* from L^*?

ARAA.

Annu. Rev. Astron. Astrophys.

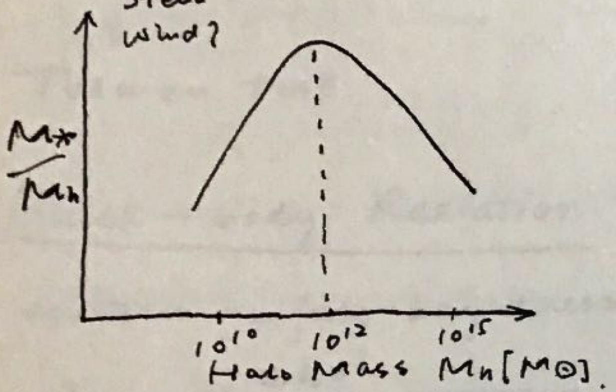
Galaxy ~ tracer of DM.

← physical models empirical model →

HD models semi-analytic model

simulated halo & gas

Stella wind? supernovae? AGN?



~~[illegible]~~

distance modulus:

$$M - m = 2.5\log\left(\frac{d_{L,0}}{d_L}\right)^2 = 5\log\left(\frac{d_{L,0}}{d_L}\right)$$

$d_{L,0} = 10\,\mathrm{Mpc}$.

$m = M + 5\log d_L + 25$.

if we use $d_L = (1+z)\,r_1 \approx \frac{c}{H_0}\left[z + \frac{1}{2}(1-q_0)z^2 + \cdots\right]$.

$$m = M - 5\log H_0 + 5\log[z + \cdots + 25$$

Type Ia Supernovae.

Binary Stars $\left\{\begin{array}{l}\text{white dwarf} \\ \text{Companion star}\end{array}\right\} \xrightarrow{\text{Accrete}} 1.44M_\odot \longrightarrow$ explode

↓

constant M.

$M_B = 0.8(\Delta m_{15} - 1.1) - 19.5 \rightarrow$ standard candle.

Parameter Estimation

$(\Omega_{m,0}, \Omega_{\Lambda,0}, M)$.

define parameter $\theta \equiv (\Omega_{m,0}, \Omega_{\Lambda,0}, M)$.

maximuming the posterior probability (liklihood).

$$L(\theta) = \exp\left(-\frac{1}{2}\chi^2\right).$$

$$\chi^2 = \sum_{i=1}^{m} \left(\frac{m(z_i;\theta) - m_i}{\sigma_{m_i m}}\right)^2.$$

we are interested in $\Omega_{m,0}$ and $\Omega_{\Lambda,0}$. so let

$$L(\Omega_{m,0}, \Omega_{\Lambda,0}) = \int dM \, L(\Omega_{m,0}, \Omega_{\Lambda,0}, M).$$

We can also use ~~a~~ other methods like cluster. BAO. SNe. CMB. to ~~indent~~ quantify the parameters.

Alternative Explanation $\rightarrow \Lambda$ (cosmological dimming).

$\downarrow$

dimming of the supernovae maximum light.

① Evolution? $\begin{cases} \text{young galaxy} \\ \text{old galaxy} \end{cases}$

② Interstellar dust? $\begin{cases} \text{dimming} \\ \text{red} \end{cases}$

③ <u>Grey dust</u> dim but not redden??

astrophysical dimming. (always dimming, contrast to observation).

Dark Energy.

$$\rho_\Lambda \equiv \frac{\Lambda}{8\pi G} \quad (c=1).$$

fluid eq. $\dot{\rho} = -3(\rho + p)\frac{\dot{a}}{a}$

$$p_\Lambda = -\rho_\Lambda$$

Let $p_i = \omega_i \rho_i$

$$\frac{\rho_i}{\rho_i} = -3(1+\omega_i)\frac{\dot{a}}{a}$$

$$\ln\rho_i \propto \ln a \cdot [-3(1+\omega_i)\cancel{\tfrac{\dot{a}}{a}}].$$

$\rho_i \propto a^{-n_i} \quad n_i = 3(1+\omega_i).$

$$\Omega_i \equiv \frac{\rho_i}{\rho_c} = \frac{8\pi G}{3H^2}\rho_i$$

$$\frac{\Omega_i}{\Omega_j} = \frac{\rho_i}{\rho_j} \propto \frac{a^{-n_i}}{a^{-n_j}} = a^{-(n_i-n_j)}.$$

Dust $\rho \propto a^{-3} \quad n_i = 3.$

zero pressure. $\omega_i = 0.$

Radiation $\rho_M \propto a^{-4}. \quad n_i = 4.$

$\omega = \frac{1}{3}.$

$\Lambda. \quad n_i = 0 \quad \omega_i = -1.$

curvature. $\Omega_k \equiv -\frac{k}{aH^2} = \frac{8\pi G}{3H^2}\cdot\rho_k$

$\therefore \rho_k = -\frac{3}{8\pi G}\cdot\frac{k}{a^2} \Rightarrow n_i = 2 \quad \omega_i = -\frac{1}{3}.$

	w_i	n_i
matter	0	3
~~dust~~ radiation	$\frac{1}{3}$	4
curvature	$-\frac{1}{3}$	2
vacuum	-1	0.

Deceleration parameter.

$$q(t) = -\frac{1}{H^2}\frac{\ddot{a}}{a} = -a\frac{\ddot{a}}{\dot{a}^2}$$

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3p)$$

$$q = \frac{1}{H^2}\frac{4\pi G}{3}(\rho + 3p)$$

$$= \frac{1}{2\rho_c}(\rho + 3p).$$

$$q(t) = \sum_i \frac{1}{2\rho_c}[\rho_i + 3p_i].$$

$$= \sum_i \frac{1}{2\rho_c}[\rho_i + 3w_i\rho_i].$$

$$= \sum_i \frac{1}{2}\Omega_i[1 + 3w_i].$$

$$= \sum_i \frac{1}{2}\Omega_i[1 + n_i - 3].$$

$$= \sum_i \frac{n_i - 2}{2}\Omega_i$$

component $n_i > 2$. $\rightarrow$ deceleration

$n_i < 2$ $\rightarrow$ acceleration.

Empty Universe.

$\Omega_k = 1$

$\Rightarrow n=2 \quad q=0.$

expands linearly with time.

$\frac{\Omega_i}{\Omega_\Lambda} \propto \frac{1}{a^{n_i}}$ $a \uparrow$ (universe expands) $\frac{\Omega_i}{\Omega_\Lambda} \downarrow$

WFIRST $\rightarrow \Lambda$.

Thermal History of Universe

CMB

$B_\lambda(T)\,[\mathrm{erg\cdot s^{-1}\cdot cm^{-2}\,\mathring{A}^{-1}\,sr^{-1}}] = \frac{2hc^2/\lambda^5}{e^{\frac{hc}{\lambda k_B T}}-1}$

$B_\nu(T)\,[\,\cdots\ \mathrm{Hz^{-1}}\,] = \frac{2h\nu^3/c^2}{e^{h\nu/k_B T}-1}$

total energy density. $u = aT^4$

$a = \frac{4\sigma}{c} = 7.57\times10^{-15}\ \mathrm{erg\cdot cm^{-3}\cdot K^{-4}}$

photon: $\langle E \rangle = 2.70\,k_B T$.

Today # density of CMB photons.

$n_{\gamma,0} = \frac{u}{\langle E \rangle} = \frac{a}{2.70\,k_B} T^3.$

$= 4.1\times10^2\ \mathrm{cm^{-3}}.$

average density of baryons.

$n_{b,0} = \frac{\Omega_{b0}\,\rho_{critic}}{\langle m_b \rangle} = 2.5\times10^{-7}\ \mathrm{cm^{-3}}.$

$n_{b,0}/n_{\gamma,0} = 6.1\times10^{-10}.$

energy densities.

$$\frac{u_{b,0}}{u_{r,0}} = \frac{n_b}{n_\gamma} \cdot \frac{\langle m_b \rangle c^2}{\langle u \rangle} = 9.0 \times 10^2 .$$

non-relativistic dark matter and relativistic neutrinos.

$$\frac{u_{m,0}}{u_{rad,0}} = 3.4 \times 10^3 .$$

$$\rho_m = \rho_0 (1+z)^3$$

$$\rho_\gamma = \rho_{\gamma,0} (1+z)^4 .$$

$z \approx 3340$ ~~$\rho_m = \rho_\gamma$~~. $u_m / u_\gamma = 1$.

$$T_{CMB} = 2.73 (1+z) \text{ K} . \approx 3000 \text{ K} .$$

Thermal History.

Very early universe. radiation dominated.

$$a(t) = \left(\frac{t}{t_0}\right)^{1/2} .$$

$$T = 2.73 (1+z) \text{ K} .$$

$$\Rightarrow T \propto a^{-1}$$

$$\Rightarrow t \propto a^{2} \propto T^{-2} .$$

$$t(s) = \left(\frac{T}{1.5 \times 10^{10} \text{K}}\right)^{-2} = \left(\frac{T}{1.3 \text{MeV}}\right)^{-2}$$

universe at $t < 1$ s.

$T \to \infty$.

planck unit.

Schwarzschild radius

matter wave $\lambda = \frac{h}{p} = \frac{2\pi\hbar}{mc} \approx \pi r_s \rightarrow \frac{2mG}{c^2}$.

Planck mass

$$m_p = \left(\frac{\hbar c}{G}\right)^{1/2} = 10^{19}\,\text{GeV}.$$

Planck length.

$$l_p = \frac{\hbar}{m_p c} = \left(\frac{\hbar G}{c^3}\right)^{1/2} \approx 10^{-33}\,\text{cm}.$$

planck time

$$t_p = \frac{l_p}{c} = \left(\frac{\hbar G}{c^5}\right)^{1/2} \approx 10^{-43}\,\text{s}.$$

Freeze-out

rate of interaction for a given process Γ.

expansion rate of universe H.

$$\Gamma >> H.$$

$$t_c = \frac{1}{\Gamma} << t_H = \frac{1}{H}$$

$\Rightarrow$ local thermal equilibrium is established.

$t_c \sim t_H$: the particles in question decoupled from thermal plasma.

$t \approx 10^{-[illegible]}\,\text{s}.$

Grand Unified Theory (GUT) era.

$t \approx 10^{-35} s$ $\quad T \sim 10^{27} K \approx 10^{14} GeV$

the GUT transition

electronweak and strong forces emerge

① inflation $\quad t = 10^{-36} s \rightarrow 10^{-24} s$.

(to explain flat universe, the isotropic ~~univer~~ CMB / causality).

$t \sim 10^{-12} s$ $\quad T = 10^{15} K \approx 100 \text{ GeV}$.

the electron-weak transition.

$t \sim 10^{-6} s$ $\quad T \sim 10^{12-13} K \approx 200 \text{MeV} - 1 \text{GeV}$. ($m_p \approx 938.3 \text{MeV}$)

QCD. Quark-~~Hadron~~Hadron transition.

$t \sim 10^{-6} \rightarrow t \approx$ a few seconds.

Lepton era.

decoupling of neutrinos.

$t \approx 1 s$ $\quad T \approx 1 \text{MeV} \approx 10^{10} K$. $\rightarrow \frac{\Gamma_\nu}{H} \approx \left(\frac{T}{1.6 \times 10^{10} K} \right)^3$.

($T < 10^{10} K$, decouple).

$T \approx 10^{12} K$. electrons. neutrinos.

anti-particles & photons

equilibrium $\quad e^{\pm} + \gamma \leftrightarrow e^{\pm} + \gamma$.

$$e^+ + e^- \leftrightarrow \gamma + \gamma$$

$$\nu + \bar{\nu} \leftrightarrow e^+ + e^-$$

$$\nu + e^{\pm} \leftrightarrow \nu + e^{\pm}$$

CMB $\quad$ 372,000 yrs $\longrightarrow 1 s$.

$t \approx 5s \quad T < 500\,keV \approx 5\times10^{9}K$

$m_e = 511$ ~~MeV~~ keV. one path reaction. $e^+ + e^- \to 2\gamma$.

T

$T_\nu = \left(\frac{4}{11}\right)^{1/3} T_\gamma$

↳ cosmic neutrino background.

$T_{\nu,0} \approx 1.95K$.

information behind CMB. $\sim 10^{-5}K$ fluctuation.

Primordial Nucleosynthesis.

BBN - Big Bang Nucleosynthesis

Neutron - Proton Ratio. $(t < 1s,\ T > 10^{10}K)$.

$n + e^+ \longleftrightarrow p + \bar{\nu}_e$

$p + e^- \longleftrightarrow n + \nu_e$

$n \leftrightarrow p + e^- + \bar{\nu}_e$.

equilibrium $\Rightarrow$ Boltzmann distribution

$$\left(\frac{n}{p}\right)_{eq} = e^{-\frac{\Delta mc^2}{K_B T}} = e^{-\frac{1.5}{T_{10}}}$$

$\Delta m = m_n - m_p = 1.29\,MeV$

$= 1.5\times10^{10}K$.

T_{10}: in units of $10^{10}K$.

$$\frac{\Gamma}{H} = \left(\frac{T}{1.6 \times 10^{10} K} \right)^3$$

$$\frac{H(z)}{H_0} = \sqrt{\Omega_{r,0}(1+z)^4 + \cdots}$$

$u = \rho c^2 \propto T^4$.

$\rightarrow H \propto T^2$.

$\Gamma \propto n(\nu_e, \bar{\nu}_e) \propto T^3$.

$\propto \langle \sigma \rangle \rightarrow T^2$.

$\Rightarrow \frac{\Gamma}{H} \propto T^3$.

$k_B T_d \approx 0.8 \text{ MeV} = (m_n - m_p) - m_e$

$t \approx 2.6 s$ ~~n~~ neutron decouple.

$$\frac{n}{p} = \exp\left[-\frac{\Delta m c^2}{k_B T} \right] = \exp\left[-\frac{1.3}{0.8} \right] = 0.2 = \frac{1}{5}.$$

But neutron will decay if not combined into the ~~n~~ nuclei

<u>Deuterium Formation</u> $n \rightarrow D \rightarrow {}^3He \rightarrow {}^4He$

$n \rightarrow p + e^- + \bar{\nu}_e \quad \tau_n = 880.3 \pm 1$ second.

$p + n \leftrightarrow d + \gamma$. energy difference 2.225 MeV.

(too small for photons in Rayleigh-Jeans tail).

$\Rightarrow$ Deuterium bottleneck.

$T_D = 8\times 10^8 K$. $t \approx 300s$.

Deuterium can survive.

$$\frac{n}{p}\Big|_{t=300} = \frac{n}{p}\Big|_{t\approx 2.6} \exp[-300/880.3]$$

$$= 0.71 \times \frac{1}{5}$$

$$\approx 0.14 \approx 1:7.$$

Primordial helium mass fraction Y_p.

$$Y_p = \frac{4\cdot\frac{n}{2}}{\cancel{4\cdot\frac{n}{2}}\ n+p} = \frac{2n}{8n} = 0.25.$$

$$\frac{^4He}{H} = \frac{1}{12} \text{ by number.}$$

$$\eta \equiv \frac{n_b}{n_\gamma}$$

$$\eta_{10} = 273.3\,\Omega_{b,0} h^2 \left(\frac{2.7255\,K}{T_{\gamma,0}}\right)^3.$$

? Stellar nucleosynthesis.

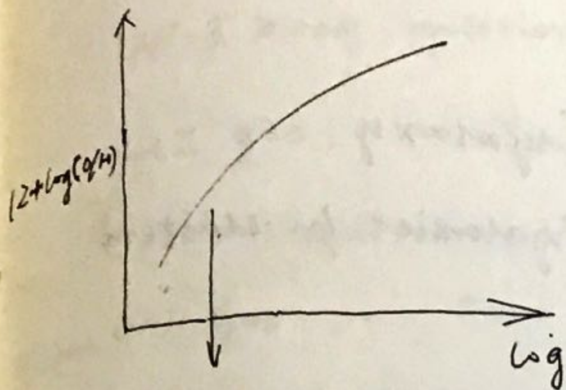
first star $z \sim 20$.

metal heavier than helium.

stellar composition.

H 0.74 (X)
He 0.24 (Y) (sun) $\longrightarrow$ (for sun, $Y_p < 0.24$).
Metal 0.02 (Z).

Mass - metallicity relation



Dwarf galaxy → Y_p more close to initial universe.

if we use $Y_p = 0.245 \pm 0.004 \Rightarrow \Omega_{b,0} \in [0.018, 0.059]$.
$(h = 0.675)$.

D/H ratio

absorption line of deuterium in AGN.

$100\,\Omega_{b,0} h^2 = 2.235 \pm 0.05$.

^{7}Lithium.

we use $^7Li/H = (1.6 \pm 0.3) \times 10^{-10}$.

if $\Omega_{b,0} = 4.83 \times 10^{-2}$ BBN predicts) 3 times?

missing Lithium problem.

(from CMB $\Omega_{b,0}^2 h^2 = 2.226 \times 10^{-2}$).

CMB also gives $\Omega_{m,0} = 0.312$.

$\Rightarrow \dfrac{\Omega_{b,0}}{\Omega_{m,0}} = 15.6\%$.

non-baryonic and a baryonic dark matter problem.

non-baryonic.

① rotation curve of spiral galaxy.

②. velocity dispersion of galaxies in clusters.

③ Large Scale Structure.

④. gravitational lensing.

baryonic "dark matter".

ISM. IGM. warm gas. $\sim 10^3 K$.

Beyond the Standard Model.

dark matter → dark radiation?

→ relativistic component?

total ~~neutrin~~ energy density.

$$u = aT^2 \left[1 + \frac{7}{4} + \frac{7}{8} N_\nu\right].$$

↳ more kinds of neutrino?

propbably 3.

Coronal gas → HII gas → Warm HI

→ Cool HI → Diffuse H_2 → Dense H_2

~~→~~ Cool Stellar outflows.

Coronal gas: $\gtrsim 10^{5.5}$ K

UV. & X ray emission; Radio synchrotron emission.

HII gas: 10^4 K.

Optical emission; Thermal radio continuum.

Warm HI gas: 5000 K

HI 21 cm emission, absorption.

Optical & UV. absorption lines.

Cool HI gas: ~100 K

HI 21 cm emission, absorption.

Optical & UV ~~ap~~ absorption lines.

Diffuse H_2 ~50 K

HI 21 cm emission, absorption.

CO 2.6 mm emission

optical, UV absorption lines.

Dense H_2 10~50 K

non-baryonic dark matter.

$\Omega_{m,0} \approx \Omega_{0,0} \cdot 6$.

Mass to light ratio.

B - Band. 3800Å - 4800Å

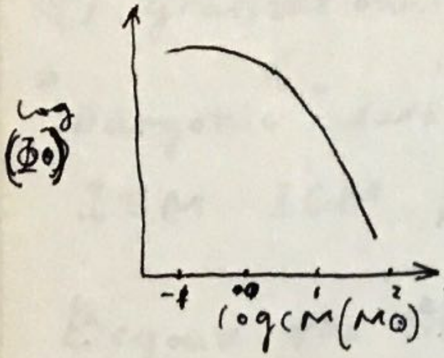
$M_\odot / L_{\odot,B}$

$L_{MW,B} \approx 2.3 \times 10^{10} L_{\odot,B}$.

if every stars in milky way are sun-like.

$\Rightarrow M_{MW} = 2.3 \times 10^{10} M_\odot$. naive prediction!

We have to consider
Initial Mass Function (IMF).



in galaxy

dynamic mass

= stellar + gas (DM in galaxy can be ignored).

massive stars.

$M/L_B \sim 10^{-3} M_\odot / L_{\odot,B}$.

low mass stars.

$M/L_B \sim 10^{3} M_\odot / L_{\odot,B}$.

$L \propto M^{3.5}$. (from IMF).

within 1 kpc from the sun.

$\langle M/L_B \rangle \approx 4 M_\odot / L_{\odot,B}$.

↳ S.I units. 1.7×10^5 kg/W.

Luminosity Function (LF).

total stellar luminosity density in B-Band.

$$j_{stars,B} = 1.1\times10^{8} L_{\odot,B}\ Mpc^{-3}.$$

If we assume $\langle M/L_B \rangle$ in 1 kpc from sun is ~~uniform~~ universal. we can estimate that:

$$\rho_c = j_{stars,B} \cdot \langle M/L_B \rangle = 4.4\times10^{8} M_\odot/Mpc^3$$

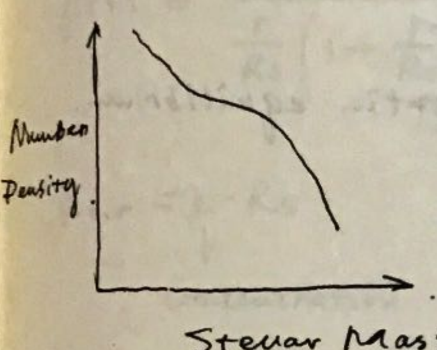
$$\rho_c \approx 1.36\times10^{11} M_\odot\ Mpc^{-3}.$$

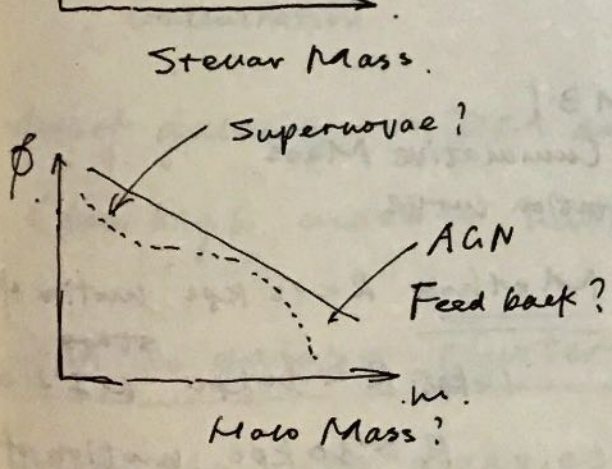
$$\Rightarrow \Omega_{star} \approx 3\times10^{-3}.$$

Mass Function

galaxy luminosity $\longrightarrow M_{star}$.

SED 拟合.





Rotation curve?

$$v(R) = \frac{v_r(R) - v_{gal}}{\sin i} = \frac{v_r(R) - v_{gal}}{\sqrt{1 - b^2/a^2}}$$

inclination angle

$$\frac{v^2}{R} = G\frac{M(R)}{R^2}.$$

$$\Rightarrow v(R) = \sqrt{\frac{GM(R)}{R}}.$$

assume density is a constant.

$M = \frac{4}{3}\pi R^3 \rho.$

$v(R) \propto R.$

$r >> R.$ $M(R)$ is constant.

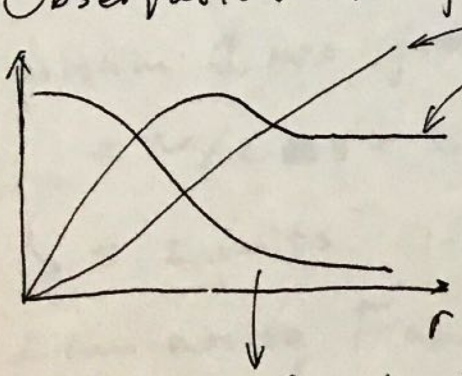
$v(R) \propto R^{-1/2}$ Keplerian rotation.

matter distribution in hydrostatic equilibrium.

$\rho \propto \frac{1}{R^2}$ $dM = 4\pi \rho R^2 dR \propto dR.$

$\Rightarrow v(R) = \text{const}.$

Observation ~~A~~ of M31.



Method $R < 10$ kpc motion of stars.

$10\text{kpc} \le R < 30\text{kpc}$ HI 21 cm.

$R > 30$ kpc motion of satellite galaxy

Stellar light $I(R) = I(0)\exp\left(-\frac{R}{R_s}\right).$ M31 $R_s \approx$

Milky Way.

$$M(R)_{MW} = 9.6\times10^{10} M_\odot \left(\frac{v}{220\,km/s}\right)^2 \left(\frac{R}{8.5\,kpc}\right).$$

mass to light ratio.

$$\langle M/L_B\rangle \approx 50 M_\odot / L_{\odot,B} \left(\frac{R_{halo}}{100\,kpc}\right).$$

$$M^{MW}_{dyn} \approx 1\times10^{12} M_\odot.$$

DM Halo profile.

Navarro-Frenk-White profile.

NFW profile.

$$\rho(r) = \frac{\rho_0}{\frac{r}{R_s}\left(1+\frac{r}{R_s}\right)^2}$$

R_s → scale radius.

$R_{vir} = c \cdot R_s$.

c → concentration.

dwarf galaxy → DM dominate.

(high mass to luminosity ratio).

DM in galaxy clusters

More than 100 galaxies. $10^{14} M_\odot \sim 10^{15} M_\odot$

(upto 1000).

Dynamics are dominated by DM.

relaxed" cluster → equilibrium.

star formation. → galaxy ↑

~~DM~~ halo acceration → DM halo ↑

$$\left.\begin{array}{l} \text{DM} \\ \text{galaxies} \\ \text{intrainstella gas.} \end{array}\right\} \rightarrow \text{hard to observe.}$$

Virial Theorem.

$-2\langle K\rangle = \langle U\rangle$.

K: kinetic energy.

U: potential energy.

$$K = \sum_i \frac{1}{2} m_i |\dot{x}_i|^2$$

$$= \frac{1}{2} M \langle v^2\rangle$$

$$M \downarrow \sum_i m_i$$

$$U = -\frac{1}{2} G \sum_{\substack{i,j \\ i\neq j}} \frac{m_i m_j}{|\vec{x}_j - \vec{x}_i|}$$

$$U = -\alpha \frac{GM^2}{r_h}$$ r_h: half mass radius

α density profile of the cluster. typically $\alpha = 0.4$

$$M\langle v\rangle^2 = \alpha \frac{GM^2}{r_h}.$$

$$M = \frac{\langle v\rangle^2 r_h}{\alpha G}.$$

coma cluster.

$\langle z \rangle \approx 0.0232$

$\langle v_r \rangle = c\langle z \rangle = 6955\ \mathrm{km/s}$.

$H_0 = 67.5\ \mathrm{km/s}$.

$D = 103\ \mathrm{Mpc}$.

one dimensional velocity dispersion.

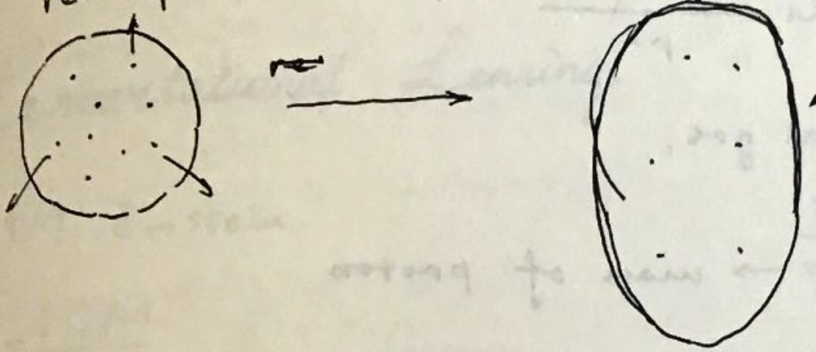
$\sigma_r = \langle (v_r - \langle v_r \rangle)^2 \rangle^{1/2} \approx 880\ \mathrm{km/s}$.

3D dispersion. $r_h = 1.5\ \mathrm{Mpc}$.

$\langle v^2 \rangle = 3 \times 880\ \mathrm{km/s}$. $\Rightarrow M_{vir}^{coma} = \dfrac{\langle v^2 \rangle r_h}{\alpha G} \approx 2 \times 10^{15} M_\odot$.

real space

redshift space.

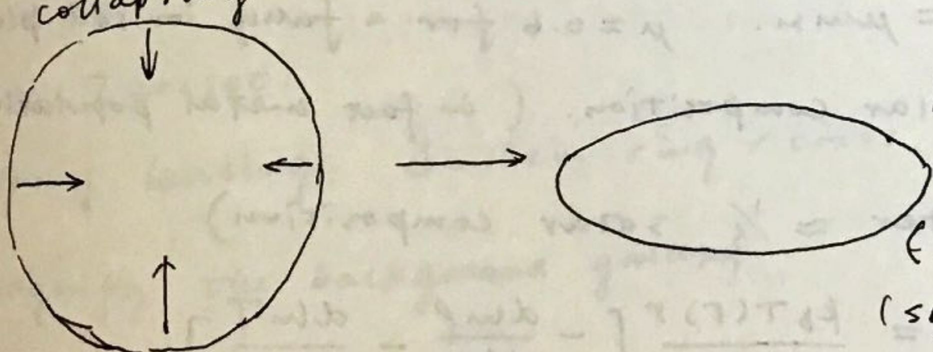


finger of God

Redshift space Distortion.

RSD

large structure in collapsing



Kaiser effect.

(Pancake)

(squashing effect)

Gaussian Random field?

$M_{vir} \sim 1.7 \times 10^{15} M_\odot \left(\dfrac{\sigma_r}{1000\ \mathrm{km/s}}\right)^2 \left(\dfrac{r_h}{1\ \mathrm{Mpc}}\right)$.

$M_{stars}^{coma} \approx 3 \times 10^{13} M_\odot \sim 2\%$.

X-ray → intrastella gas. ~10%.

$$M_{gas}^{coma} \approx 2\times10^{14} M_\odot.$$

$$\left\langle \frac{M}{L_B} \right\rangle \approx \frac{2\times10^{15} M_\odot}{8\times10^{12} L_{\odot,B}} = \frac{250 M_\odot}{L_{\odot,B}} \sim 5 \text{ times of MW}.$$

too hot for gas to cool down and form stars.

Hydrostatic Equilibrium

$$\frac{dp}{dr} = -G\frac{\overbrace{M(r)}^{\text{total mass inside } r.}\,\rho(r)}{r^2}.$$

For an ideal gas:

$$p = \frac{\rho k_B T}{\mu m_p}$$

m_p → mass of proton

μ → average molecular weight.

$\langle m \rangle = \mu m_H$. $\mu \simeq 0.6$ for a fully ionised plasma for solar composition. (in fact metal population in cluster ≈ 1/3 solar composition).

$$M(r) = \frac{k_B T(r)\, r}{G\mu m_p}\left[-\frac{d\ln\rho}{d\ln r} - \frac{d\ln T}{d\ln r}\right].$$

$$M_{hydro}^{Coma} = (1-2)\times10^{15} M_\odot.$$

Typical Cluster. $M/L_{B,\odot} \sim 250 M_\odot / L_{B,\odot}$.

Sungaev-Zel'rorich effect.

SZ effect.

CMB photons scattered by cluster free electrons.

$$\frac{\Delta I_\nu^{RJC}}{I_\nu^{RJ}} = -2 \int \frac{k_B T}{m_e c^2} \sigma_T n_e \, dl$$

(method to find cluster!).

↳ $f_b \approx 10 \sim 12\%$ baryon ~~free electron~~ fraction.

Gravitational Lensing

1919 Einstein.

$$\alpha = \frac{4GM}{c^2 b}$$

$b = R_\odot$

$\alpha \approx 1.7$ arcsec.

① strong lensing: Einstein ring / cross.

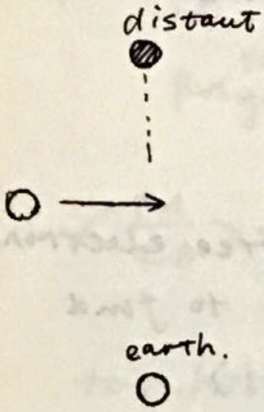
magnify the background galaxy.

② Weak lensing

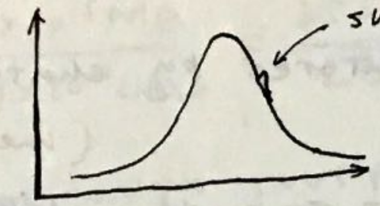
light from distant galaxy deflected by large-scale structure: Cosmic Shear.

distortion < 2%. (shape of distant galaxy).

③ Microlensing.



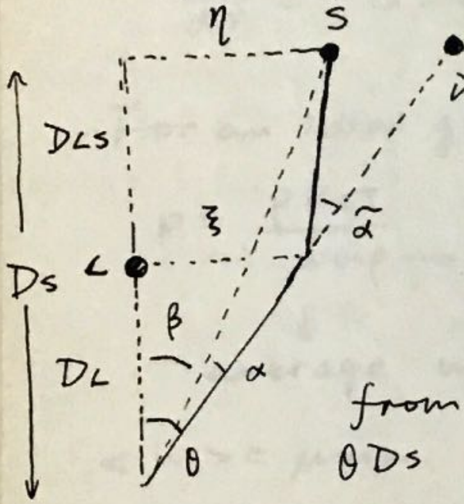
light curve.



small blip if a planet is moving around the star.

thin lense approximation.

impact parameter $\xi\,(b) \gg \frac{2GM}{c^2}$ (R_s, Schwarzschild radius)



in Schwarzschild metric

deflection angle

$$\tilde{\alpha}(\xi) = \frac{4GM(\xi)}{c^2}\frac{1}{\xi}$$

from geometry.

$$\theta D_S = \beta D_S + \tilde{\alpha} D_{LS}.$$

$$\alpha(\theta) = \frac{D_{LS}}{D_S}\tilde{\alpha}(\theta).$$

$$\Rightarrow \beta = \theta - \alpha(\theta).$$

Einstein Radius.

$$\xi = D_L\theta.$$

$$\beta(\theta) = \theta - \frac{D_{LS}}{D_L D_S}\frac{4GM}{c^2\theta}.$$

$\beta = 0$

Def $\theta_E = \left(\frac{4GM}{c^2} \cdot \frac{D_{LS}}{D_L D_S} \right)^{1/2} = \left(\frac{2 R_S D_{LS}}{D_L D_S} \right)^{1/2}$.

$\rightarrow$ ring like image.

$\beta \leq \theta_E$. $\rightarrow$ strong magnification.

$\beta \gg \theta_E$ $\rightarrow$ very little magnification.

$$\frac{\theta}{\text{arcsec}} = \left(\frac{M}{10^{11.09} M_\odot} \right)^{1/2} \left(\frac{D_L D_S / D_{LS}}{\text{Gpc}} \right)^{-1/2}.$$

galaxy to galaxy lensing: $\theta_E \sim$ order of arcsec.

(source) (lense)
galaxy − cluster lensing $\theta_E \sim$ order of 10s arcsec

microlensing of stars.

$D_{LS}/D \simeq 1/2$ in MW.

$$\theta_E = 0.64 \times 10^{-3} \text{ arcsec} \left(\frac{M}{M_\odot} \right)^{1/2} \left(\frac{D_L}{10 \text{ kpc}} \right)^{-1/2}.$$

milliarcsec. hard to resolve.

Image Position and Magnification.

$$\beta = \theta - \frac{\theta_E^2}{\theta} \quad \text{if } \beta \neq 0.$$

$$\theta_{1,2} = \frac{1}{2} \left(\beta \pm \sqrt{\beta^2 + 4\theta_E^2} \right)$$

total flux $\uparrow$
surface brightness $\rightarrow$.

magnification: ratio between the solid angles subterded by the image and the source.

$$\mu = \frac{\theta d\theta}{\beta d\beta}.$$

or

$\mu \equiv \det M = \frac{1}{\det A}$ $M = A^{-1}$ magnification tensor.

$\mu_{1,2} = \left(1 - \left[\frac{\theta_E}{\theta_{1,2}}\right]^4\right)^{-1} = \frac{u^2+2}{2u\sqrt{u^2+4}} \pm \frac{1}{2}$. $u = \frac{\beta}{\theta_E}$.

$\beta \to \infty$. $u \to \infty$.

$\begin{cases} \mu_+ = 1 \\ \mu_- = 0. \end{cases}$ $\mu = |\mu_+| + |\mu_-| = 1$.

$\beta \to 0$, $u \to 0$.

$\mu \to \infty$. Einstein ring → magnification diverge.

$\beta = \theta_E$.. $u = 1$, $\mu = 1.17 + 0.17 \approx 1.34$.

I_+

$\theta < \theta_E$.

magnification is negative.
↓
mirror inverted.

$\mu = |\mu_1| + |\mu_2| = \frac{u^2+2}{u\sqrt{u^2+4}} \geq 1$

$\mu_1 - \mu_2 = 1$

Singular Isothermal Sphere. (SIS).

treating galaxy as "gas" of stars with pressure. $p = \frac{\rho k_B T}{m}$ T "Temperature" (moving stars)
m: stella mass

$\Rightarrow m\sigma^2 = k_B T$.

Spherical distribution of stars and gas.

$$\rho(r) = \frac{\sigma_v^2}{2\pi G} \cdot \frac{1}{r^2}$$

(singularity at $r=0$).

near the center has a finite core.

$$\rho = \frac{\rho_c}{1+\left(\frac{r}{r_0}\right)^2} \quad r_0 \text{ (core radius)}.$$

$r \gg r_0 \rightarrow$ SIS. $\quad 3D \xrightarrow{\text{projection}} 2D$.

circularly symmetric surface.

mass distribution.

$$\Sigma(\xi) = \frac{\sigma_v^2}{2G} \cdot \frac{1}{\xi}$$

$$M(\xi) = \int_0^\xi \Sigma(\xi') 2\pi \xi' d\xi' = \frac{\pi \sigma_v^2}{G} \xi$$

$$\downarrow$$

$$\alpha(\xi) = \frac{4GM(\xi)}{c^2} \frac{1}{\xi}$$

$$\Rightarrow \tilde{\alpha}(\xi) = \frac{4\pi}{c^2} \sigma_v^2 = 1.4'' \left(\frac{\sigma_v}{220\,\text{km/s}}\right).$$

for cored model.

$$\tilde{\alpha}(\xi) = \frac{4\pi}{c^2} \sigma_v \frac{\xi}{(\xi_c^2 + \xi^2)^{1/2}}.$$

Lensing Model?

~~Re~~ Dark Matter Revisited.

neutrino?

(ν_e, ν_μ, ν_τ). mass eigenvalues (ν_1, ν_2, ν_3)

mass difference:

$\Delta m_{32}^2 = (2.4 \pm 0.1) \times 10^{-3}\ eV^2$.

$\Delta m_{21}^2 = (7.5 \pm 0.2) \times 10^{-5}\ eV^2$.

matter power spectrum.

Planck CMB. $\sum m_\nu < 0.18\,eV$?

Neutrino and photon number ratio?

number density of each neutrino species.

$\sim 3/11$ CMB photons.

$n_{\nu,0} = 3 \times \left(\frac{3}{11}\right) n_{\gamma,0} = 3.35 \times 10^2\ cm^{-3}$.

$\Omega_{\nu,0} h^2 = \frac{\sum m_\nu}{93.14\,eV}$

use $h = 0.675$.

$\Omega_{\nu,0} < 0.004$.

Cold Dark Matter. ΛCDM model.

(non-relativistic).

WIMPS (Weak Interacting Massive Particles).

$m_\nu > 10\ GeV$.

supersymmetry (SUSY) theorem.

Modified physics

law of gravity is modified on very large scale.

MOND. modified Newtonian dynamics.

in the regime of weak acceleration.

$F = ma^2/a_0 \quad a_0 \approx 1 \times 10^{-8} cm/s^2$.

BBN ends at $t = 300s$.

$T \simeq 8 \times 10^8 K$.

In the universe, we have:

photons, protons, helium nuclei, electron.

DM particles & neutrino.

Thomson scattering. $\gamma + e^- \rightarrow \gamma + e^-$

cross section.

$\sigma_T = 6.6 \times 10^{-25} cm^2$ photon & electron in equilibrium

proton: ~~coulomb~~ coulomb interaction.

polarization.

~~inte~~ Scattering rate per photon $\Gamma_{T,e}$.

mean free path: $\lambda = \frac{1}{n_e \sigma_T}$

$\Gamma_{T,e} = \frac{c}{\lambda} = n_e \sigma_T c$.

$\frac{\Omega_{b,0} \rho_{crit}}{\langle m_b \rangle}$

↑

When the Universe is fully ionized $n_e \simeq n_b = n_{b,0}(1+z)^3$

$\Gamma_{T,e} \approx 2.5 \times 10^{-7} \times 6.6 \times 10^{-25} \times 3 \times 10^{10} (1+z)^3 \, s^{-1}$

$\approx 5 \times 10^{-21} (1+z)^3 \, s^{-1}$

$$H(z) = H_0\sqrt{\Omega_{b,0}(1+z)^3 + \Omega_{rad,0}(1+z)^4 + \Omega_{k,0}(1+z)^2 + \Omega_{\Lambda,0}}.$$

transition from radiation dominated to matter dominated.

$$\Omega_{b,0}(1+z)^3 = \Omega_{rad,0}(1+z)^4.$$

$\Rightarrow z \simeq 3380.$ ($\Omega_{rad,0} = 9\times10^{-5}$, photon & neutrino).

$$T_{eq} = 2.7255 \times 3381 = 9215\,K.$$

Recombination.

ions (proton, He^{2+}, e^-).

$$\Gamma_{T,e} < H.$$

$$H + \gamma \underset{\text{recombination}}{\overset{\text{photoionization}}{\rightleftharpoons}} p + e^-.$$

number density n_x of particles with mass m_x.

$$n_x = g_x\left(\frac{m k_B T}{2\pi\hbar^2}\right)^{3/2}\exp\left(-\frac{m_x c^2}{k_B T}\right) \qquad k_B T \ll m_x c^2.$$

g_x: statistical weight of particle . x.

H atoms. protons & free electrons.

$$\frac{n_H}{n_p n_e} = \frac{g_H}{g_p g_e}\left(\frac{m_H}{m_p m_e}\right)^{3/2}\left(\frac{k_B T}{2\pi\hbar}\right)^{-3/2}\exp\left[\frac{(m_p+m_e-m_H)}{k_B T}c^2\right]$$

simplifications.

(1) $g_H/g_p g_e = 1$

(2) $m_H \simeq m_p$

(3) ionisation potential $Q = m_p + m_e - m_H$.

Saha eq:

$$\frac{n_H}{n_p n_e} = \left(\frac{m_e k_B T}{2\pi \hbar^2} \right)^{-3/2} \exp\left(\frac{Q}{k_B T} \right).$$

ionization fraction.

$$X = \frac{n_p}{n_p + n_H} = \frac{n_e}{n_b}.$$

Define

$X = 0.5$ when the ionization happen.

$$n_H = \frac{1-X}{X} n_p, \quad n_e = n_p.$$

$$\Rightarrow \frac{1-X}{X} = n_p \left(\frac{m_e k_B T}{2\pi \hbar^2} \right)^{-3/2} \exp\left(\frac{Q}{k_B T} \right).$$

baryon to photon ratio. $\eta = \frac{n_b}{n_\gamma} = \frac{n_p}{X n_\gamma}.$

black body spectrum.

$$n_\gamma = \frac{2.404}{\pi^2} \left(\frac{k_B T}{\hbar c} \right)^3 = 0.244 \left(\frac{k_B T}{\hbar c} \right)^3.$$

$$\Rightarrow n_p = 0.244 X \eta \left(\frac{k_B T}{\hbar c} \right)^3.$$

$$\frac{1-X}{X^2} = 3.84 \eta \left(\frac{k_B T}{m_e c^2} \right)^{3/2} \exp\left(\frac{Q}{k_B T} \right) \equiv S(T, \eta).$$

$$X = \frac{-1 + \sqrt{1+4S}}{2S}.$$

$k_B T \gg Q$. $\exp \to 1$. S is small

$$X = \frac{-1 + (1+4S)^{1/2}}{2S} \to 1.$$

if $X = 0.5$. $\eta = 6.1 \times 10^{-10}$.

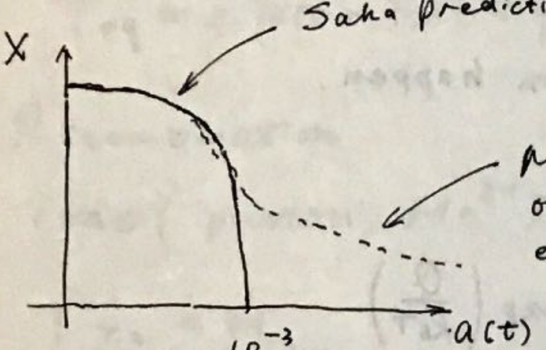
$k_B T_{rec} = 0.323\,eV \approx \frac{Q}{42}$ $Q \rightarrow 13.6\,eV$.

$T_{CMB,0} = 2.7255\,K$.

$T_{rec\infty} = 0.323\,eV = 3750\,K$.

$z_{rec} \approx 1375$

$t_{rec} \approx 251000\,yr$.



Why?

i) $X = 0.9 \rightarrow X = 0.1$ $\Delta t \sim 70000\,yr$.

ii) overionized.

Lyman α photon resonantly scattering.

$p + e^- \rightarrow H + \gamma$

until:
two photons emission from $2s \rightarrow 1s$.

(no longer resonant photons).

$z_{dec} = 1090$.

$T_{dec} = 2971\,K$

$t_{dec} = 372000\,yr$.

last scattering surface (layer).

Time line.

	z	$T(K)$	t (Myr)
Radiation-Matter Equality	3380	9215	0.047.
Recombination	1375	3750	0.251
Last scattering / Photon decoupling	1090	2971	0.372.

Cosmic Microwave Background

today $\langle h\nu \rangle = 6.3 \times 10^{-4}$ eV.

COBE. WMAP. Planck.

Olbers' paradox $\lambda = 1.1$ mm.

$T(\theta, \phi)$ Observation.

$$\langle T \rangle = \frac{1}{4\pi} \int T(\theta,\phi) \sin\theta \, d\theta \, d\phi = 2.7255 \pm 0.0006 \, K.$$

variation ~30mK

$$\frac{\delta T}{T}(\theta,\phi) = \frac{T(\theta,\phi) - \langle T \rangle}{\langle T \rangle} \rightarrow \left\langle \left(\frac{\delta T}{T} \right)^2 \right\rangle^{1/2} = 1.1 \times 10^{-5}$$

(remove the dipole at 10^{-3} caused by earth's relative motion).

Why there's so small variation?

comoving horizon distance at time t.

$$\omega_H = \int_0^t \frac{c\,dt}{a(t)} = \int_0^R \frac{c\,dR}{R^2 H(R)} \qquad H(R) \approx H_0$$

$$\approx 2 \frac{c}{H_0} \Omega_{m,0}^{-1/2} (1+z)^{-1/2}.$$

$$d_H = (1+z)^{-1} \omega_H = 2 \frac{c}{H_0} \Omega_{m,0}^{-1/2} (1+z)^{-3/2}$$

$$\delta\theta = \frac{d_H}{D} = \frac{d_H}{d_A} = \frac{d_H}{\frac{1}{1+z}\int_0^z \frac{c\,dz}{H_0\sqrt{\Omega_{m,0}(1+z)^3+\Omega_{\Lambda,0}}}}.$$

Matting relation assume $\Omega_{\Lambda,0}=0$

$$d_A = 2\frac{c}{H_0}\cdot\frac{1}{\Omega_{m,0}^2(1+z)^2}\left[\Omega_{m,0}z+(\Omega_{m,0}-2)\left(\sqrt{(1+\Omega_{m,0}z}-1\right)\right].$$

$z \gg 1$

$$\approx 2\frac{c}{H_0}\frac{1}{\Omega_m z}$$

$$\delta\theta_{horizon} \approx \left(\frac{\Omega_{m,0}}{z_{dec}}\right)^{1/2} \sim 1^\circ.$$

concordance Universe.

$\delta\theta \approx 1.8^\circ.\ \propto \Omega_{m,0}^{-0.1}$

$\Rightarrow$ horizon problem? Why CMB is homogenus?

$\Rightarrow$ "Inflation" (solution?).

10^{30} times. at $t=10^{-35}$ s $T=10^{27}$ K $\sim 10^{14}$ GeV.

Large Scale Structure. $\rightarrow$ CMB initial condition.

fluctuation. (Message In CMB).

$T\uparrow$ Density $\uparrow$. DM collapse than baryon.

$\rightarrow$ dense $\rightarrow$ star.

UV sheilding?

$z \simeq 6$ fully ionized (Small amount molecule gas in galaxy - star formation).

$z = 1 \sim 6$. evolution. of galaxy and planet.

$m - \sigma$ relation. Mass of super Massive BH in the galaxy ~ Mass of the ~~centre of~~ core of galaxy

$\Rightarrow$ ~~co~~evolution of BH and galaxy.

CMB & Temperature fluctuation.

① primary fluctuation.

at z_{rec}

② secondary fluctuation.

optical depth (scattering during $z_{rec} \to z \sim 0$).

③ Terotiary flucturation.

dust and gas in our Galaxy.

(polarization).

$$I_\nu \text{ (intensity)} = \frac{2h\nu^3/c^2}{e^{h\nu/k_B T} - 1}$$

$\Rightarrow I_\nu \propto \nu^2 T$ when $h\nu \ll k_B T$ (Rayleigh - Jeans Tail)

$$\Rightarrow \frac{\delta I_\nu}{I_\nu} = \frac{\delta T}{T}$$

Spherical harmonics.

Temperature fluctuation $\to$ correlation function.

$$C(\theta) = \left\langle \frac{\delta T}{T}(\vec{r}) \cdot \frac{\delta T}{T}(\vec{r}') \right\rangle_{\vec{r} \cdot \vec{r}' = \cos\theta}.$$

Power Spectrum.

$$\Delta_T^2 = \frac{l(l+1)}{2\pi} C_l \langle T\rangle^2 \cdot (\mu K)^2.$$

$l \Leftrightarrow \theta$.

$l=0$ monopole $\langle T \rangle$.

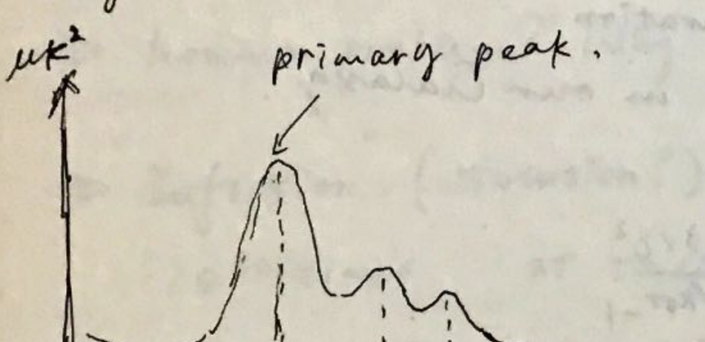
peculiar velocity. (doppler).

$l=1$ dipole motion of the earth. $\langle (\delta T/T)^2 \rangle^{1/2} \sim 10^{-3} K$.

$l \geq 2$. Higher Multipole. ✓

0.31 ± 0.05. Ω_m (How much mass is needed to produce the dipole)

angular coherent.



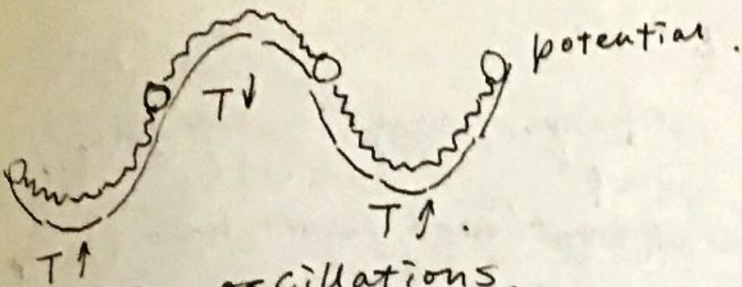
photons. protons. electrons He nuclei. neutrino. DM.

baryon - photon fluid.

energy density is dominated by radiation.

($\Omega_b \approx 1/6\, \Omega_m$).

competition of the gravitational potential & radiation pressure.



acoustic oscillations.

$u = aT^4$.

The first doppler peak (sound horizon).

size of horizon at z_{dec}.

for photon: $s_{hor,}(z_{dec}) = 2\frac{c}{H_0}\Omega_{m,0}^{-1/2}(1+z_{dec})^{-3/2}$.

sound wave travelling at speed c_s.

$c_s^2 = \frac{dp}{d\rho}$. $\quad p = \cancel{\tfrac{1}{3}}\, w\rho c^2$. $\quad w = 1/3$

$\Rightarrow c_s = \frac{1}{\sqrt{3}}c$.

Sound horizon: $\frac{2}{\sqrt{3}}\frac{c}{H_0}\Omega_{m,0}^{-1/2}(1+z_{dec})^{-3/2}$.

Mattig Relation $d_A(z) \approx 2\frac{c}{H_0}\frac{1}{\Omega_{m,0} z}$

$\theta_{hor,s} = \frac{1}{\sqrt{3}}\left(\frac{1-\Omega_{k,0}}{z_{dec}}\right)^{1/2}$. $\quad \theta \sim \Omega_{k,0}$.

$\theta_{hor,s} \approx \cancel{N8}\, 1^\circ \propto \Omega_{m,0}^{-0.1}$ first peak → curvature.

not sensitive to Ω_m.

Baryon loading → $\Omega_{b,0}$.

$\Omega_{b,0}h^2 = (2.226 \pm 0.016) \times 10^{-2}$.

$h = 0.675 \Rightarrow (4.884 \pm 0.035) \times 10^{-2}$.

BBN $\Rightarrow (4.83 \pm 0.10) \times 10^{-2}$.

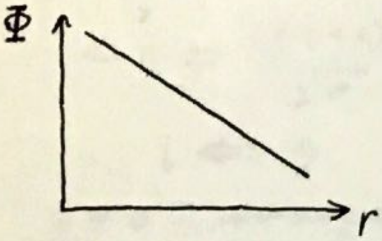
⇒ stronger compression.

⇒ higher ~~single~~ odd peak.

damping ~~whic~~ tail → Ω_m.

IMF

Initial Mass function.



diffusive galaxy (dwarf galaxy).

↑

different dark matter halo. ?

Ghostly galaxy may be missing dark matter.

- formation of galaxy?
- existence of dark matter.

problems:

only 10 examples. (fitting?).